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Introduction

A purpose of statistical mechanics is to understand physical properties of macroscopic systems based on laws governing the behavior of individual particles at the microscopic scale ([8], p. 3, [49], p. 1). Classical physic's *microstate* is an ensemble of positions and momenta of individual particles forming the whole system, while the *macrostate* is described by macroscopic quantities, like i.e. the energy which we focus on below ([49], p. 3). A model used for understanding macroscopic rules of thermal energy transport is a system of oscillators arranged on an integer lattice with Hamiltonian dynamics. It is necessary to emphasize the fact that the nonlinearity of such model is the key element in this context. A way of mathematically capturing it is turning to probabilistic methods and considering Hamiltonian dynamics of oscillators which is linear (harmonic oscillators) but stochastically perturbed, so chaotic behavior owing to nonlinearity in deterministic system is replaced by the perturbation. Such models have been intensively examined in recent years, see i.e. [9, 10, 11, 12, 13, 14, 28, 35, 36, 38, 39]. In this elaboration we write about some results. Our aim is to get familiar, at an elementary level, with some probabilistic tools of description and investigation of such systems. We review some stochastic processes and related concepts, and we get some insight into how they are used to describe physical processes involved. To get a first sketch of mathematical model, we consider a set of oscillators indexed by $y \in \mathbb{Z}^d$ ([13], p. 188). We can visualize a lattice of points in space, having integer coordinates, with oscillating particles arranged on them, so they form a crystalline structure. Oscillator attached to a site y is at any given time t characterized by the pair

$(\mathfrak{p}_y, \mathfrak{q}_y)$ of its momentum $\mathfrak{p}_y = \mathfrak{p}_y(t)$ and position $\mathfrak{q}_y = \mathfrak{q}_y(t)$. Position $\mathfrak{q}_y$ is measured relative to the equilibrium position of the oscillator. We denote

$$\mathfrak{p} := \{\mathfrak{p}_y\} \quad \text{and} \quad \mathfrak{q} := \{\mathfrak{q}_y\}.$$

Deterministic Hamiltonian of the system has the form

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \sum_{y \in \mathbb{Z}^d} \mathfrak{e}_y,$$

where $\mathfrak{e}_y$ is total energy of oscillator at site y given by

$$\mathfrak{e}_y := \frac{1}{2}\mathfrak{p}_y^2 + W(\mathfrak{q}_y) + \frac{1}{2} \sum_{|y-y'|=1} V(\mathfrak{q}_y - \mathfrak{q}_{y'}). \tag{1}$$

Here V and W are potentials. Potential W, called *pinning potential*, is responsible for the interaction between the oscillator and the system as a whole, while potential V defines interaction between neighboring oscillators, and depends only upon their relative position. Deterministic dynamics is governed by the system of equations

$$\begin{cases} \dot{\mathfrak{q}}_y = \partial_{\mathfrak{p}_y} \mathcal{H}(\mathfrak{p}, \mathfrak{q}), \\ \dot{\mathfrak{p}}_y = -\partial_{\mathfrak{q}_y} \mathcal{H}(\mathfrak{p}, \mathfrak{q}). \end{cases} \tag{2}$$

If potentials are quadratic, then the system (2) is linear and we deal with *harmonic oscillators*. In this case, energy transport in vibrating lattice is ballistic. Superdiffusive or diffusive transport is a result of nonlinearity ([10] p. 1, [11] p. 68-69). Introduction of a stochastic perturbation instead leads to nondeterministic models. System with quadratic potential presented in i.e. [13] is weakly perturbed as follows

$$\begin{cases} \dot{\mathfrak{q}}_y = \partial_{\mathfrak{p}_y} \mathcal{H}(\mathfrak{p}, \mathfrak{q}) \\ \dot{\mathfrak{p}}_y = -\partial_{\mathfrak{q}_y} \mathcal{H}(\mathfrak{p}, \mathfrak{q}) + \dot{\xi}_y^{(\epsilon)}[\mathfrak{p}]. \end{cases} \tag{3}$$

The stochastic term $\xi_y^{(\epsilon)}[\mathfrak{p}]$, $y \in \mathbb{Z}^d$, is of order ϵ, where $0 < \epsilon \ll 1$. The perturbation is defined in such way, that it preserves total energy of the chain.

This work consists of four chapters.

- *Chapter 1* contains a review of certain stochastic processes and related notions, used in construction of described models. In particular, it touches on topics of Lévy and Gaussian processes, compound Poisson process, Gaussian measures and Gaussian fields, stochastic integration and differentiation.

- *Chapter 2* presents one–dimensional model of stochastically perturbed harmonic oscillators, together with a different scale perspectives on the energy distribution and its evolution in time:

  - microscopic: the wave function and the Wigner transform;
  - hyperbolic scaling limit: solution of the linear phonon Boltzmann equation;
  - superdiffusive and diffusive scaling limit: solution of the (fractional) heat equation.

- *Chapter 3* is devoted to the model with Gaussian noise introduced in [10, 11], and outlines some theorems presented and used in proofs of asymptotics for this model in source articles [13], [28] and [38].

- *Chapter 4* describes model with Ornstein–Uhlenbeck perturbation introduced in [39] and sketches some calculations used there in the proof of asymptotics.

Chapter 1

Probabilistic tools – an overview

1.1 Fourier transform

Schwartz space of functions $f : \mathbb{R} \to \mathbb{C}$, denoted by $\mathcal{S}(\mathbb{R})$, is defined as the space of all complex valued functions on $\mathbb{R}$ having derivatives of any order (infinitely differentiable) and satisfying

$$\|f\|_{n,m} := \sup_{x\in\mathbb{R}} \left(1+x^2\right)^{n/2} \sum_{i=0}^{m} \left|f^{(i)}(x)\right| < \infty \quad \text{for all} \quad n,m \in \mathbb{N}_0.$$

Members of the Schwartz space are referred to as *rapidly decreasing functions*. Given functions f_n, $n \in \mathbb{N}$ and f belonging to $\mathcal{S}(\mathbb{R})$ we say that the sequence $\{f_n : n \in \mathbb{N}\}$ is convergent to f in $\mathcal{S}$ if it is convergent to f in all norms $\|\cdot\|_{n,m}$.

The space of all complex–valued continuous linear functionals on $\mathcal{S}(\mathbb{R})$ is denoted by $\mathcal{S}'(\mathbb{R})$, wherein we say that a functional ϕ is continuous if $\phi(f_n) \to \phi(f)$ given that $f_n \to f$ in $\mathcal{S}$. Members of $\mathcal{S}'(\mathbb{R})$ are called *tempered distributions*. Every function $f \in \mathcal{S}$ is identified with a distribution ϕ_f given by

$$\phi_f(g) = \int_{\mathbb{R}} f(x)g(x)dx, \qquad g \in \mathcal{S}(\mathbb{R}), \tag{4}$$

so $\mathcal{S}(\mathbb{R})$ is embedded in $\mathcal{S}'(\mathbb{R})$. Given $\phi \in \mathcal{S}'(\mathbb{R})$ we define n–th order *derivative* $\phi^{(n)}$ as the distribution satisfying

$$\phi^{(n)}(f) = (-1)^n \phi\big(f^{(n)}\big) \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}).$$

Given $f \in \mathcal{S}(\mathbb{R})$, the *Fourier transform* $\mathcal{F}f$ of f is defined as a function $\widehat{f}$ on $\mathbb{R}$ given by the following formula

$$\widehat{f}(p) = \mathcal{F}f(p) := \int_{\mathbb{R}} \mathrm{e}^{-2\pi i p x} f(x) dx.$$

The *inverse Fourier transform* of $f \in \mathcal{S}(\mathbb{R})$ is defined as

$$\widetilde{f}(x) = \mathcal{F}^{-1}f(x) := \int_{\mathbb{R}} \mathrm{e}^{2\pi i p x} f(p) dp.$$

Theorem 1 ([26], p. 40 and p. 42)**.** *Let $f \in \mathcal{S}(\mathbb{R})$. Then*

(i) $\mathcal{F}f \in \mathcal{S}(\mathbb{R})$, $\mathcal{F}^{-1}f \in \mathcal{S}(\mathbb{R})$ and both mappings $\mathcal{F}$, $\mathcal{F}^{-1}$ are bijective on $\mathcal{S}(\mathbb{R})$,

(ii) $\mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}\mathcal{F}^{-1}f = f$,

(iii) the Fourier transform of n–th order derivative $f^{(n)}$ is equal to $(2\pi i p)^n \widehat{f}(p)$,

(iv) the Fourier transform of $x^n f(x)$ is equal to $i^n\,(2\pi)^{-n}\,\widehat{f}^{(n)}(p)$.

For any $\phi \in \mathcal{S}'(\mathbb{R})$ the *Fourier transform* $\mathcal{F}\phi$ is defined as a distribution $\widehat{\phi}$ such that

$$\widehat{\phi}(f) = \phi(\widehat{f}\,) \quad \text{for every} \quad f \in \mathcal{S}(\mathbb{R}).$$

Analogously, the *inverse Fourier transform* $\mathcal{F}^{-1}\phi$ is defined as the distribution $\widetilde{\phi}$ satisfying

$$\widetilde{\phi}(f) = \phi(\widetilde{f}\,) \quad \text{for every} \quad f \in \mathcal{S}(\mathbb{R}).$$

For ϕ_f given by (4) we have $\widehat{\phi}_f = \phi_{\widehat{f}}$ and $\widetilde{\phi}_f = \phi_{\widetilde{f}}$.

Theorem 2 ([26], p. 47)**.** *Let $\phi \in \mathcal{S}'(\mathbb{R})$.*

(i) Both mappings $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bijective on $\mathcal{S}'(\mathbb{R})$,

(ii) $\mathcal{F}^{-1}\mathcal{F}\phi = \mathcal{F}\mathcal{F}^{-1}\phi = \phi$,

(iii) the Fourier transform of n–th order derivative $\phi^{(n)}$ of ϕ is equal to $(2\pi i p)^n \widehat{\phi}$,

(iv) the Fourier transform of $x^n\phi$ is equal to $i^n\,(2\pi)^{-n}\,\widehat{\phi}^{(n)}$.

For any pair f, g belonging to the Banach space $L_2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$ the following identity holds ([26], p. 51)

$$\int_{\mathbb{R}} f(x)g^*(x)dx = \int_{\mathbb{R}} \widehat{f}(p)\widehat{g}^*(p)dp, \tag{5}$$

here g^* denotes the complex conjugate of g. In particular $\mathcal{F}$ is an isometry on L_2

$$\int |f(x)|^2 dx = \int |\widehat{f}(p)|^2 dp.$$

Let us now introduce discrete Fourier transform of a sequence $\{f(z) : z \in \mathbb{Z}\}$. We denote by $\mathbb{T}$ the one–dimensional torus represented by interval $[-1/2, 1/2]$ with endpoints $-1/2$ and $1/2$ identified. For a complex function $z \mapsto f(z)$ defined on integers z and belonging to $l_2(\mathbb{Z})$ we define $\mathcal{F} : f \mapsto \widehat{f}$ by the formula

$$\widehat{f}(k) := \sum_{z\in\mathbb{Z}} e^{-2\pi i k z} f(z), \qquad k \in \mathbb{T}. \tag{6}$$

The analogue of equality (5) reads

$$\sum_{z\in\mathbb{Z}} f(z)g^*(z) = \int_{\mathbb{T}} \widehat{f}(k)\widehat{g}^*(k)dk.$$

The inverse Fourier transform $\mathcal{F}^{-1} : u \mapsto \widetilde{u},\ u \in L_2(\mathbb{T})$, is defined as

$$\widetilde{u}(z) := \int_{\mathbb{T}} \mathrm{e}^{2\pi i z k} u(k)dk.$$

1.2 Gaussian random variables and random fields

We use the following definition of Gaussian random variable ([27], p. 3).

Definition 3 (Gaussian random variable)**.** *A real valued random variable X is called Gaussian if it has the characteristic function $\phi_X(p) = \mathbb{E}e^{ipX}$, $p \in \mathbb{R}$, of the form*

$$\phi_X(p) = e^{i\mathrm{m}p - \frac{1}{2}\sigma^2 p^2} \tag{7}$$

where $\mathrm{m} \in \mathbb{R}$ *and* $\sigma^2 \geq 0$.

The parameter m is the mean value of X and σ^2 is the variance. If $\mathrm{m} = 0$ then we say that X is *centered* or *symmetric* Gaussian. We note that this definition admits $\sigma^2 = 0$, so a function X with $\mathbb{P}[X = \mathrm{m}] = 1$ is also Gaussian. If $\sigma^2 > 0$, then the law of X with characteristic function (7) has the density function

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{(x-\mathrm{m})^2}{2\sigma^2}\right\}. \tag{8}$$

Distribution with the density (8) is called Gaussian or *normal* and if it is the law of random variable X then we write $X \sim \mathcal{N}(\mathrm{m}, \sigma^2)$.

Definition 4 (Jointly Gaussian random variables)**.** *A finite collection $X_1, X_2, \ldots, X_n$ of Gaussian random variables is called jointly Gaussian if for arbitrary real numbers $t_1, t_2, \ldots, t_n$ random variable $\sum_{i=1}^n t_i X_i$ is Gaussian.*

Assume that T is an arbitrary set. A real–valued function $(t, s) \mapsto \mathrm{R}(t, s)$ on $T \times T$ is called *nonnegative definite*, if for every $n \in \mathbb{N}$, every n elements $t_1, \ldots, t_n$ of T and every n real numbers $c_1, \ldots, c_n$ we have

$$\sum_{i=1}^n \sum_{j=1}^n \mathrm{R}(t_i, t_j) c_i c_j \geq 0.$$

Definition 5 (Gaussian random field)**.** *Given an arbitrary set T, a Gaussian random field in T is a collection of random variables $\{X(t) : t \in T\}$ on common probability space, such that for every $n \in \mathbb{N}$ and $t_1, \ldots, t_n$ of T, the random variables*

$$X(t_1), \ldots, X(t_n)$$

are jointly Gaussian.

It follows that if $\{X(t) : t \in T\}$ is a Gaussian random field then function $\mathrm{R}(t, s)$ defined as

$$\mathrm{R}(t, s) := \mathbb{E}\big(X(t) - \mathbb{E}X(t)\big)\big(X(s) - \mathbb{E}X(s)\big) \tag{9}$$

is nonnegative definite. On the other hand, for every nonnegative definite function $\mathrm{R}(\cdot, \cdot)$ on T and every $\mathbb{R}$–valued function $t \mapsto \mathrm{m}(t)$ there exists a Gaussian random field on T such that (9) holds with $\mathbb{E}X(t) = \mathrm{m}(t)$ (see [2], p. 5).

Definition 6 (Stationary field)**.** *Assume that T is an additive group. A Gaussian random field on T is called* stationary*, if $\mathrm{m} = \mathbb{E}X(t)$ does not depend on $t \in T$ and covariance $\mathrm{R}(t, s)$ depends only on the difference $t - s$, i.e. $\mathrm{R}(t, s) = \widetilde{\mathrm{R}}(t - s)$ for some $\widetilde{\mathrm{R}} : T \to \mathbb{R}$.*

1.3 Gaussian measures on Banach spaces

Let us denote by H a real separable Hilbert space. We say that a bounded linear operator Q on H is

- *symmetric*, if for every pair $h_1, h_2 \in H$ we have $\langle h_1, Qh_2 \rangle = \langle Qh_1, h_2 \rangle$;
- *nonnegative*, if for every $h \in H$ we have $\langle h, Qh \rangle \geq 0$;
- *positive*, if for every $h \in H$, $h \neq 0$ we have $\langle h, Qh \rangle > 0$.

Nonnegative bounded linear operator Q on H is of *trace class* (see general definition in [41] on p. 330), if for a complete orthonormal system $\{e_n : n \in \mathbb{N}\}$ in H

$$\operatorname{Tr} Q := \sum_{n=1}^{\infty} \langle Q e_n, e_n \rangle < \infty.$$

$\operatorname{Tr} Q$ is called the *trace* of Q and its value is independent of the choice of orthonormal basis $\{e_n : n \in \mathbb{N}\}$ ([41], p. 333).

A probability measure μ defined on the family of Borel subsets $\mathcal{B}(H)$ of H with a scalar product $\langle \cdot, \cdot \rangle$ is called *Gaussian*, if for arbitrary element $h \in H$, the $\mathbb{R}$–valued random variable $\langle \cdot, h \rangle$ on probability space $(H, \mathcal{B}(H), \mu)$ is Gaussian. It follows that in this case ([20], p. 53)

– there exist an element $\mathrm{m} \in H$, such that

$$\int_H \langle x, h \rangle \mu(dx) = \langle \mathrm{m}, h \rangle, \quad h \in H, \quad \text{and}$$

– there exists a symmetric, nonnegative and trace class operator Q on H such that

$$\int_H \langle x, h_1 \rangle \langle x, h_2 \rangle \mu(dx) - \langle \mathrm{m}, h_1 \rangle \langle \mathrm{m}, h_2 \rangle = \langle Q h_1, h_2 \rangle, \quad h_1, h_2 \in H.$$

The element m is referred to as the *mean* of the measure μ and operator Q is called *covariance operator* of μ. We write $\mu \sim \mathcal{N}(\mathrm{m}, Q)$. The characteristic function of μ, defined as $\widehat{\mu}(h) := \int_H \exp\{i \langle x, h \rangle\} \mu(dx)$, $h \in H$, is

$$\widehat{\mu}(h) = \exp\left\{ i \langle \mathrm{m}, h \rangle - \frac{1}{2} \langle Q h, h \rangle \right\}.$$

A random variable X with values in H is called Gaussian, if its distribution is Gaussian.

Proposition 7 ([20], p. 57)**.** *For arbitrary positive, symmetric trace class operator Q on H, and arbitrary $\mathrm{m} \in H$, there exists a Gaussian measure μ on H with mean m and covariance Q.*

Proof. (A sketch). There exists a complete orthonormal basis $\{f_n : n \in \mathbb{N}\}$ in H consisting of eigenvectors of Q ([41], p. 316). Denote eigenvalue of f_n by λ_n, for $n \in \mathbb{N}$. By positivity of Q, eigenvalues are positive. We also have

$$\operatorname{Tr} Q = \sum_{n=1}^{\infty} \langle Q f_n, f_n \rangle = \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Let $X^{(n)}, n \in \mathbb{N}$ be a sequence of independent identically distributed real valued Gaussian random variables, $X_1 \sim \mathcal{N}(0,1)$, on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define H–valued random variables $\xi_n, n \in \mathbb{N}$ by

$$\xi_n := \sum_{k=1}^{n} \sqrt{\lambda_k}\, X_k f_k.$$

We have $\mathbb{E}\xi_n = 0$, $\mathbb{E}\|\xi_n\|^2 = \sum_{k=1}^{n} \lambda_k < Tr\, Q < \infty$, and for every $h_1, h_2 \in H$

$$\mathbb{E}\langle \xi_n, h_1 \rangle \langle \xi_n, h_2 \rangle = \sum_{k=1}^{n} \langle Q h_1, f_k \rangle \langle f_k, h_2 \rangle.$$

The sequence of ξ_n is convergent in $L_2(\Omega; H)$, and thus there exists a H–valued random variable ξ such that

$$\xi = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, X_k f_k \qquad \mathbb{P}\text{– almost surely.}$$

In particular, for every $h \in H$, $\langle \xi, h \rangle$ is $L_2(\Omega; \mathbb{R})$–limit of $\langle \xi_n, h \rangle$, so it is a Gaussian mean zero random variable ([27], p. 4, Theorem 1.3). For $h_1, h_2 \in H$, the covariance of $\langle \xi, h_1 \rangle$ and $\langle \xi, h_2 \rangle$ equals

$$\mathbb{E}\langle \xi, h_1 \rangle \langle \xi, h_2 \rangle = \sum_{k=1}^{\infty} \langle Q h_1, f_k \rangle \langle f_k, h_2 \rangle = \langle Q h_1, h_2 \rangle.$$

The measure μ is the law of random variable $\mathrm{m} + \xi$.

□

Assume that B is a real separable Banach space and H is a real separable Hilbert space continuously and densely embedded in B, that is, there exists a linear continuous and injective mapping $\iota : H \to B$ such that the image ιH is dense in B. Denote by B^* the dual space of B, and by H^* the dual space of H. For $x \in B$, $\phi \in B^*$ we denote $\langle x, \phi \rangle_B := \phi(x)$. Let $\iota^* : B^* \to H^*$ be the dual mapping of ι defined by the relation

$$\langle \iota\xi, \phi \rangle_B = \langle \xi, \iota^*\phi \rangle_H, \qquad \xi \in H, \, \phi \in B^*.$$

Let μ be a probability measure on the Borel sets of B such that for every $\phi \in B^*$, the random variable $\langle \cdot, \phi \rangle_B$ on probability space $(B, \mathcal{B}(B), \mu)$ is Gaussian with mean zero and variance $\|\iota^*\phi\|^2_{H^*}$, i.e.

$$\int_B \exp\{i\langle x, \phi \rangle_B\} \, \mu(dx) = \exp\Big\{ -\frac{1}{2}\|\iota^*\phi\|^2_{H^*}\Big\}, \quad \phi \in B^*, \tag{10}$$

here $\|\cdot\|_{H^*}$ is the norm in H^*. Then the measure μ is by definition a *Gaussian measure on* B and the triple (B, H, μ) is called *abstract Wiener space*. The Hilbert space H is referred to as *reproducing kernel Hilbert space* for μ ([50], [53]).

1.4 Gaussian processes

We start this section with the Kolmogorov theorem about continuity and regularity of paths of stochastic processes (see i.e. [20] on p. 73). It involves the notion of a *modification* of stochastic process $\{X(t) : t \geq 0\}$, which is, by definition, every such process $\{\widetilde{X}(t) : t \geq 0\}$ that

$$\mathbb{P}[X(t) \neq \widetilde{X}(t)] = 0, \qquad \text{for all } \ t \geq 0.$$

Theorem 8 (Kolmogorov test)**.** *Let $X = \{X(t) : t \geq 0\}$ be a stochastic process with values in a separable Banach space with norm $\|\cdot\|$. Assume that there exist constants $\delta > 1$, $\epsilon > 0$ and $M > 0$ for which*

$$\mathbb{E}\|X(t) - X(s)\|^\delta \leq M|t-s|^{1+\epsilon}$$

for every pair $t, s \in [0, T]$. *Then there exists a modification* $\widetilde{X}$ *of* X *on* $[0, T]$ *such that, for arbitrary* $\gamma < \epsilon/\delta$, *the paths* $t \to \widetilde{X}(t)$ *are almost surely Hölder continuous with exponent* γ. *Precisely, for some positive constant* C *depending on* γ *almost surely we have*

$$\|\widetilde{X}(t) - \widetilde{X}(s)\| \leq C|t - s|^{\gamma}, \quad \textit{for all } t, s \in [0, T].$$

We say that a stochastic process $\{X(t) : t \geq 0\}$

(i) has *stationary increments* if distributions of $X(t+h) - X(h)$ and $X(t) - X(0)$ are the same for every $h > 0$;

(ii) has *independent increments* if the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent for every $t_0 < t_1 < \cdots < t_n$.

Let $\{X(t) : t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $t \geq 0$ let $\mathcal{F}_t$ be a σ–field, $\mathcal{F}_t \subseteq \mathcal{F}$. The family $\{\mathcal{F}_t : t \geq 0\}$ is called a *filtration* if $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s < t$. A stochastic process $\{X(t) : t \geq 0\}$ is *adapted* to a filtration $\{\mathcal{F}_t : t \geq 0\}$ if $X(t)$ is $\mathcal{F}_t$–measurable for every t. The *natural filtration* of process $\{X(t)\}$ consists of the σ–fields

$$\mathcal{F}_t := \sigma\{X(s) : s \leq t\}.$$

A stochastic process $\{X(t) : t \geq 0\}$ with values in a separable Hilbert space H is called Gaussian, if for an arbitrary finite sequence of nonnegative numbers $t_1, \ldots, t_n$, the random variable $(X(t_1), \ldots, X(t_n))$ in H^n is Gaussian ([20] p. 83).

Let a bounded linear operator Q on H be symmetric, positive and of trace class. A process $W = \{W(t) : t \geq 0\}$ on H is called *Q–Wiener process* ([20], p. 86), if

(i) $W(0) = 0$ and the paths $t \mapsto W(t, \omega)$ are continuous with probability 1,

(ii) W has independent increments, and

(iii) the law of $W(t+s) - W(s)$ is $\mathcal{N}(0, tQ)$ for every $t, s \geq 0$.

A special case is Q–Wiener process in $\mathbb{R}^d$, where $Q = \{q_{ij}\}_{1 \leq i,j \leq d}$ is a positive definite symmetric matrix. In yet more special case, when Q is $d \times d$ identity matrix, then the Q–Wiener process in $\mathbb{R}^d$ is called a *standard, d–dimensional Wiener process* or a *standard, d–dimensional Brownian motion*. For a standard Brownian motion $\{w(t) : t \geq 0\}$ we have

$$\mathbb{E} \exp \{i \langle p, w(t) - w(s) \rangle\} = \exp \left\{ -\frac{1}{2}(t-s) |p|^2 \right\}, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^d$ and $|p|^2 = \langle p, p \rangle$.

A stochastic process $\{X(t) : t \geq 0\}$ together with a filtration $\{\mathcal{F}_t : t \geq 0\}$ is called *martingale* if

(i) $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ for every $t > s \geq 0$, and

(ii) $\mathbb{E}|X(t)| < \infty$ for every $t \geq 0$.

A Brownian motion $\{w(t)\}$ together with its natural filtration is a martingale since

$$\mathbb{E}[w(t)|\mathcal{F}_s] = \mathbb{E}[w(s)|\mathcal{F}_s] + \mathbb{E}[w(t) - w(s)|\mathcal{F}_s] = w(s) + \mathbb{E}[w(t) - w(s)] = w(s).$$

Let H and K be two separable Hilbert spaces over $\mathbb{R}$. A bounded linear operator on $T : H \to K$ is called *Hilbert–Schmidt*, if

$$\sum_{n=1}^{\infty} \|Te_n\|_K^2 < \infty, \tag{12}$$

where $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis in H (see [20] Appendix C). It follows that the adjoint operator $T^* : K \to H$, given by $\langle Th, k\rangle_K = \langle h, T^*k\rangle_H$, is also Hilbert–Schmidt. Furthermore, the value of (12) is the same for every orthonormal basis $\{e_n : n \in \mathbb{N}\}$.

Let us consider a series ([20] p. 97)

$$W(t) = \sum_{n=1}^{\infty} e_n w_n(t), \qquad t \geq 0, \tag{13}$$

where $\{e_n\}$ is an orthonormal basis of H and $w_n(t)$, $n \in \mathbb{N}$ is a sequence of independent standard Brownian motions in $\mathbb{R}$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The series (13) does not converge in H, however if a separable Hilbert space H_1 is such that:

- $H \subset H_1$,
- the mapping $J : H \to H_1$ given by $Jh = h$ is Hilbert–Schmidt,

then the series is almost surely convergent in H_1 and defines a Wiener process in H_1. Furthermore, for any $h \in H$ we can define

$$h(W(t)) := \sum_{n=1}^{\infty} \langle h, e_n\rangle_H \cdot w_n(t), \qquad t \geq 0. \tag{14}$$

For $m, n \in \mathbb{N}$, $m > n$ we have

$$\mathbb{E}\Big(\sum_{k=n}^{m} \langle h, e_k\rangle w_k(t)\Big)^2 = \sum_{k=n}^{m} \langle h, e_k\rangle^2 \to 0 \quad \text{as } m, n \to \infty,$$

so the series (14) is convergent in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, and hence almost surely. The limit is Gaussian with mean zero and variance $t\|h\|_H^2$. It is easily established that the process given by (14) is $\mathbb{R}$–valued Gaussian and

$$\mathbb{E}h_1(W(t))h_2(W(s)) = (t \wedge s)\langle h_1, h_2\rangle_H, \quad h_1, h_2 \in H. \tag{15}$$

More generally, let us assume that operator T is Hilbert–Schmidt operator from H to a separable Hilbert space K over $\mathbb{R}$, and define

$$TW(t) := \sum_{n=1}^{\infty} w_n(t)\, Te_n.$$

We have

$$\mathbb{E}\Big\| \sum_{k=n}^{m} w_k(t)\, Te_k \Big\|_K^2 = \sum_{k=n}^{m} \|Te_k\|_K^2 \to 0, \qquad m, n \to \infty,$$

so the series defining $TW(t)$ is almost surely convergent in K. Furthermore (we omit subscripts K, H in scalar products and norms for clarity)

$$\mathbb{E}\langle TW(t), TW(s)\rangle = \mathbb{E} \sum_{n,m} w_n(t) w_m(s) \langle Te_n, Te_m\rangle^2$$

$$= (t \wedge s) \sum_{n} \|Te_n\|^2,$$

and for $k_1, k_2 \in K$

$$\mathbb{E}\langle TW(t), k_1\rangle\langle TW(s), k_2\rangle = \mathbb{E} \sum_{n,m} w_n(t) w_m(s) \langle Te_n, k_1\rangle\langle Te_m, k_2\rangle$$

$$= (t \wedge s) \sum_{n} \langle T^* k_1, e_n\rangle\langle e_n, T^* k_2\rangle = (t \wedge s)\langle T^* k_1, T^* k_2\rangle.$$

Let us check that $TW(t)$ has continuous paths in K. We appeal to Theorem 8. Given that $k_1, k_2 \in K$, $s < t$ we have

$$\mathbb{E}\langle TW(t) - TW(s), k_1\rangle\langle TW(t) - TW(s), k_2\rangle = (t - s)\langle T^* k_1, T^* k_2\rangle.$$

Using $\|k\|_K^2 = \sum_n \langle k, g_n\rangle^2$, where $\{g_n : n \in \mathbb{N}\}$ is an orthonormal basis in K, we calculate

$$\mathbb{E}\|TW(t) - TW(s)\|^4 = \sum_{n,m} \mathbb{E}\langle TW(t) - TW(s), g_n\rangle^2 \langle TW(t) - TW(s), g_m\rangle^2.$$

Now we bring up the fact that if $X_1, X_2, \ldots, X_n$ is a sequence of n centered jointly Gaussian random variables, and n is an even number, then

$$\mathbb{E}\prod_{k=1}^{n} X_k = \sum\prod_{m} \mathbb{E}X_{i_m}X_{j_m},$$

where the sum is over all partitions of the set $\{1, \ldots, n\}$ into disjoint pairs $\{i_m, j_m\}$ ([27] p. 11). Using this forula we get

$$\mathbb{E}\|TW(t)-TW(s)\|^4 = \sum_n \mathbb{E}\langle TW(t)-TW(s), g_n\rangle^2 \sum_m \mathbb{E}\langle TW(t)-TW(s), g_m\rangle^2$$

$$+2\sum_{n,m}\big(\mathbb{E}\langle TW(t)-TW(s), g_n\rangle\langle TW(t)-TW(s), g_m\rangle\big)^2$$

$$= (t-s)^2\big(\sum_n \|Tg_n\|^2\big)^2 + 2(t-s)^2\sum_{n,m}\langle Tg_n, Tg_m\rangle^2.$$

The last sum above satisfies $\sum_{n,m}\langle Tg_n, Tg_m\rangle^2 \leq \big(\sum_n \|Tg_n\|^2\big)^2$. In consequence we obtain

$$\mathbb{E}\|TW(t)-TW(s)\|^4 \leq 3\big(\sum_n \|Tg_n\|^2\big)^2(t-s)^2,$$

and by the Kolmogorov test we get that the process $\{TW(t) : t \geq 0\}$ has a continuous modification in K.

1.5 Infinitely divisible random variables

A function $\phi : \mathbb{R}^d \to \mathbb{C}$ is called nonnegative definite if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \phi(p_i - p_j)c_i c_j^* \geq 0$$

for every $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $p_1, \ldots, p_n \in \mathbb{R}$. The following classical result comes from Bochner (see [20] on p. 49 for it's generalized version concerning probability measures on separable Hilbert spaces).

Theorem 9 (Bochner). *A complex–valued function ϕ on $\mathbb{R}^d$ is characteristic function of a probability measure on $\mathbb{R}^d$ if and only if it is nonnegative definite, continuous and satisfies $\phi(0) = 1$.*

We say that a sequence $\{P_n : n \in \mathbb{N}\}$ of probability measures on $\mathbb{R}^d$ is convergent to a probability measure P *weakly* if $\lim_{n\to\infty} \int_{\mathbb{R}^d} f\, dP_n = \int_{\mathbb{R}^d} f\, dP$ for every bounded and continuous function f on $\mathbb{R}^d$.

Theorem 10 (see [15], p. 46). *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $\mathbb{R}^d$ having characteristic functions ϕ_n, $n \in \mathbb{N}$ and ϕ respectively. Then the weak convergence of P_n to P holds if and only if $\lim_{n\to\infty} \phi_n(p) = \phi(p)$ for every $p \in \mathbb{R}^d$.*

The *convolution* $f * g$ of two integrable and continuous functions f, g on $\mathbb{R}$ is defined as $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$, $x \in \mathbb{R}$. The *convolution* $\mu_1 * \mu_2$ of two Borel probability measures μ_1 and μ_2 on $\mathbb{R}$ is a measure given by

$$(\mu_1 * \mu_2)(A) := \int_{\mathbb{R}} \mu_1(A - x)\mu_2(dx), \quad A \in \mathcal{B}(\mathbb{R}) \tag{16}$$

where $A - x = \{y - x : y \in A\}$. We point out some properties of the convolution of measures (see [3], p. 20).

(i) Convolution of probability measures is a probability measure as well.

(ii) Convolution as a binary operation in the set of all probability measures on $\mathbb{R}$ is commutative and associative, with neutral element being Dirac δ measure,

$$\delta(A) = \mathbb{1}_A(0) = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A. \end{cases}$$

(iii) If μ_1 and μ_2 have densities p_{μ_1} and p_{μ_2}, then $\mu_1 * \mu_2$ has density $p_{\mu_1 * \mu_2}$ being convolution of the densities:

$$p_{\mu_1 * \mu_2}(x) = (p_{\mu_1} * p_{\mu_2})(x) = \int_{\mathbb{R}} p_{\mu_1}(y)p_{\mu_2}(x - y)dy.$$

(iv) If X_1, X_2 are independent random variables with probability laws μ_1 and μ_2, then $\mu_1 * \mu_2$ is the law of the sum $X_1 + X_2$. In particular, for every bounded and measurable f

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)\mu_1(dx)\mu_2(dy) = \int_{\mathbb{R}} f(x)\, \mu_1 * \mu_2(dx).$$

By $\mu^{(*n)}$ we denote μ if $n = 1$ and $\mu * \mu^{(*(n-1))}$ if $n > 1$.

A random variable X is called *infinitely divisible* if for every natural number n there exist a collection $X^{(1)}, \ldots, X^{(n)}$ of independent identically distributed random variables such that

$$X \quad \text{and} \quad \sum_{k=1}^{n} X^{(k)} \quad \text{have the same distribution.}$$

It follows that X is infinitely divisible if and only if its law μ_X has the following property: for every $n \in \mathbb{N}$ there exist a probability measure μ_n such that

$$\mu_X = \mu_n^{(*n)}.$$

A probability measure with this property is called an *infinitely divisible probability measure*. If X, Y is a pair of independent random variables, then the characteristic function $\phi_{X+Y}(\cdot)$ of sum $X + Y$ satisfies

$$\phi_{X+Y}(p) = \phi_X(p)\phi_Y(p), \quad p \in \mathbb{R}.$$

Hence characteristic function of infinitely divisible random variable X has n–th root $(\phi_X(\cdot))^{1/n}$ for every n, being also a characteristic function of a random variable. There is in fact equivalence, since every characteristic function with this property corresponds to an infinitely divisible random variable ([3], p. 23-24). We also note that a sum of two independent infinitely divisible random variables is infinitely divisible.

Every infinitely divisible random variable X is necessarily of the form

$$\phi_X(\cdot) = \exp\{\psi(\cdot)\},$$

and hence is characterized by the exponent $\psi(\cdot)$, which is called the *Lévy exponent* ([3], p. 30). Let us look at some examples.

(i) A trivial example is a deterministic random variable X such that $\mathbb{P}[X = C] = 1$, where C is some real number. In this case have

$$\phi_X(p) = \exp\{iCp\}.$$

(ii) *Poisson* random variable, usually denoted by N, with parameter $\lambda > 0$, taking values in the set of natural numbers including zero with probabilities

$$\mathbb{P}[N = n] = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}, \qquad n = 0, 1, 2\dots$$

Its characteristic function is

$$\phi_N(p) = \exp\{\lambda(\mathrm{e}^{ip} - 1)\}.$$

For every $n \in \mathbb{N}$, N can be represented as sum of n independent Poisson random variables with parameter λ/n:

$$\phi_N(p) = \left(\exp\left\{\frac{\lambda}{n}(\mathrm{e}^{ip} - 1)\right\}\right)^n.$$

(iii) Let $X^{(n)}$, $n \in \mathbb{N}$ be a sequence of independent identically distributed random variables with common distribution $\pi(\cdot)$, and let N be a Poisson random variable with parameter λ, independent of all $X^{(n)}$. The random variable

$$X = \sum_{k=1}^{N} X^{(k)},$$

called *compound Poisson*, is infinitely divisible with characteristic function

$$\phi_X(p) = \exp\Big\{\lambda \int_{\mathbb{R}} (\mathrm{e}^{ipy} - 1)\, \pi(dy)\Big\}.$$

(iv) Gaussian random variable X with mean $\mathrm{m} \in \mathbb{R}$ and variance $\sigma^2 > 0$. We recall its characteristic function

$$\phi_X(p) = \exp\left\{i\mathrm{m}p - \frac{\sigma^2 p^2}{2}\right\}.$$

1.6 Lévy processes

We say that a stochastic process $\{X(t) : t \geq 0\}$ is *stochastically continuous*, if for every constant $M > 0$ and every $s \geq 0$

$$\lim_{t \to s} \mathbb{P}[|X(t) - X(s)| > M] = 0.$$

Definition 11 (Lévy process)**.** *A stochastic process $\{X(t) : t \geq 0\}$ is called Lévy if*

(i) $X(0) = 0$ *with probability* 1*,*

(ii) $X(\cdot)$ *has stationary and independent increments,*

(iii) $X(\cdot)$ *is stochastically continuous.*

For a Lévy process, the condition of stochastic continuity is, by (ii), equivalent to the condition that $\mathbb{P}[|X(t)| > M] \to 0$ as $t \to 0^+$ ([3], p. 39).

Let $X(\cdot)$ be a Lévy process. Then for every t the random variable $X(t)$ is infinitely divisible. Indeed, for every $n \in \mathbb{N}$ we have

$$X(t) = \sum_{k=1}^{n} \Big[X\big(kt/n\big) - X\big((k-1)t/n\big) \Big],$$

and the increments under the summation sign are independent and identically distributed.

Theorem 12 ([3], p. 41)**.** *Let $\{X(t) : t \geq 0\}$ be a $\mathbb{R}^d$–valued Lévy process. For every t the characteristic function of $X(t)$ has the form*

$$\phi_{X(t)}(p) = \exp\{t\,\psi(p)\}, \qquad p \in \mathbb{R}^d, \tag{17}$$

where $\psi(\cdot)$ is the Lévy exponent of $X(1)$.

Below we describe canonical examples of $\mathbb{R}$–valued Lévy processes. We note that every example has its d–dimensional counterpart (see [3], p. 43).

– The trivial case is a deterministic motion $X(t) = \mathrm{m}t$ at constant velocity m, started at zero. It has characteristic function given by (17) with the Lévy exponent

$$\psi(p) = i\mathrm{m}p.$$

– The one–dimensional Brownian motion. It has, by definition, continuous paths, variance $\mathbb{E}X(t)^2 = \sigma^2 t$ with $\sigma > 0$ and mean $\mathrm{m}(t) = 0$, $t \geq 0$. The Lévy exponent of the Brownian motion is

$$\psi(p) = -\sigma^2 p^2/2.$$

If $\sigma = 1$, then the process is called standard (one–dimensional) Brownian motion.

– By adding exponents of two previous examples we obtain a Lévy exponent

$$\psi(p) = i\mathrm{m}p - \sigma^2 p^2/2$$

corresponding to the *Brownian motion with drift*. This is a path–continuous Gaussian process. For every $t > 0$, $X(t)$ is Gaussian with mean $\mathrm{m}t$ and variance $\mathbb{E}(X(t) - \mathrm{m}t)^2 = \sigma^2 t$.

– The *Poisson* process $N(t)$ with intensity $\lambda > 0$, which is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ with probabilities

$$\mathbb{P}[N(t) = n] = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}.$$

It has characteristic function (17) with the Lévy exponent

$$\psi(p) = \lambda(e^{ip} - 1).$$

Paths of $N(t)$ begin with value 0 and are constant on intervals between times $0 = t_0 < t_1 < t_2 < \ldots$ with jumps of size 1 at t_n, $n \in \mathbb{N}$. Hence the consecutive values of $N(\cdot)$ are $0, 1, 2$ etc. Moments $t_n, n \in \mathbb{N}$ of jumps are random and the lengths $\tau_k := t_k - t_{k-1}$ of intervals between them are independent identically distributed exponential random variables with the density

$$p_{\tau_k}(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0, \\ 0 & t < 0. \end{cases} \tag{18}$$

Number λt is the mean number of jumps up to time t, while λ^{-1} is the mean length of time interval between consecutive jumps.

– The *compound Poisson process* ([3] p. 46) with Lévy exponent

$$\psi(p) = \lambda \int_{\mathbb{R}\setminus\{0\}} (e^{ipy} - 1)\mu(dy), \tag{19}$$

where μ is a probability measure on $\mathbb{R}\setminus\{0\}$. This is a purely jump process defined as the random sum

$$X(t) = \sum_{j=1}^{N(t)} Y^{(j)}, \tag{20}$$

where $Y^{(j)}$ are independent identically distributed random variables, taking values in $\mathbb{R}\setminus\{0\}$, with the common law μ, and $N(t)$ is a Poisson process with intensity λ independent of all $Y^{(j)}$. Hence, at the time t_j of the j–th jump of $N(t)$, $X(t)$ has a jump of random size $|Y^{(j)}|$. In the special case, when $Y^{(1)}$ takes values in finite set $\{y_1, \ldots, y_n\}$, the Lévy exponent takes the form

$$\psi(p) = \sum_{k=1}^{n} (e^{ipy_k} - 1) \cdot \nu_k, \tag{21}$$

with numbers ν_k such that $N(t)$ has intensity $\lambda = \sum_{k=1}^{n} \nu_k$ and

$$\mathbb{P}[Y^{(1)} = y_k] = \frac{\nu_k}{\lambda}, \quad k = 1, \ldots, n.$$

In this case the process (20) can also be represented by the sum

$$X(t) = \sum_{k=1}^{n} y_k N^{(k)}(t), \tag{22}$$

built of independent processes $N^{(1)}(\cdot), \ldots, N^{(n)}(\cdot)$ wherein $N^{(k)}(\cdot)$ is a Poisson process with intensity ν_k for each k. Indeed, each component $(\mathrm{e}^{ipy_k} - 1) \cdot \nu_k$ of (21) is the Lévy exponent of $y_k N^{(k)}(t)$.

Now let us consider more general accumulation of Poisson processes, like infinite countable counterpart of sum (22). There are three cases.

(i) Process with Lévy exponent of the form

$$\psi(p) = \sum_{k=1}^{\infty} (\mathrm{e}^{ipy_k} - 1) \cdot \nu_k, \tag{23}$$

where all y_k are nonzero and ν_k are positive. If $\sum_{k=1}^{\infty} \nu_k < \infty$ then this is again a special case of (19). The corresponding Lévy process is defined by formula (20) given that

- $N(t)$ is Poisson with intensity $\lambda = \sum_{k=1}^{\infty} \nu_k$, independent of all $Y^{(j)}$,
- $Y^{(j)}$ are independent identically distributed with laws

$$\mathbb{P}[Y^{(1)} = y_k] = \nu_k / \lambda, \quad k \in \mathbb{N}.$$

We can also think of representation (22) with ∞ instead of n:

$$X(t) = \sum_{k=1}^{\infty} y_k N^{(k)}(t). \tag{24}$$

We sum up infinite number of independent Poisson processes $N^{(k)}(\cdot)$ – and we do not impose any conditions on the sizes of their jumps, but their intensities ν_k converge to zero so fast that, with probability 1, on every finite time interval all but finite number of processes $N^{(k)}$ are yet before their first jump. Hence the sum is finite.

(ii) If $\sum_{k=1}^{\infty} \nu_k$ is infinite, then we do not have the representation (20) anymore: neither random variables $Y^{(j)}$ nor Poisson process $N(t)$ can be defined the way we did in earlier cases. However, with further assumptions, the sum (24) can still be finite on every bounded interval. We assume that for some $\epsilon > 0$ the following conditions are satisfied

$$(a) \quad \sum_{|y_k| \geq \epsilon} \nu_k < \infty \qquad \text{and} \qquad (b) \quad \sum_{|y_k| < \epsilon} |y_k|\, \nu_k < \infty. \tag{25}$$

In (a) we add up intensities of all $N^{(k)}(t)$ such that jumps sizes $|y_k|$ are bigger than some positive number. We assume summability of the intensities, so they form process described in the point (1). Moments of jumps do not accumulate, and thus the sizes of jumps can be of arbitrary size. For the remaining group of Poisson processes, we do not assume a finite number of jumps on a finite interval. However by (b) the series

$$\sum_{|y_k| < \epsilon} (\mathrm{e}^{ipy_k} - 1) \cdot \nu_k$$

is still absolutely convergent, uniformly in p on every bounded interval:

$$\sum_{|y_k| < \epsilon} |\mathrm{e}^{ipy_k} - 1| \cdot \nu_k \leq |p| \sum_{|y_k| < \epsilon} |y_k| \nu_k < \infty,$$

and $\exp\{t\psi(p)\}$, with $\psi(p)$ defined by the series (23), is a characteristic function of a random variable. On the other hand, looking at the sum of Poisson processes itself, we see that

$$\mathbb{E}\Big| \sum_{|y_k| < \epsilon} y_k N^{(k)}(t) \Big| \leq t \sum_{|y_k| < \epsilon} |y_k| \nu_k < \infty.$$

We can first assume that all y_k are positive, and we get that $\sum_{|y_k|<\epsilon} y_k N^{(k)}(t)$ is finite with probability 1 by the monotone convergence and Tonelli's theorem. Then the finiteness of a series with both positive and negative y_k easily follows. We heuristically think that small jumps, even if there is infinite number of them on a finite interval $[0, t]$, still add up on that interval to a finite value.

(iii) If $\sum_{k=1}^{\infty} \nu_k$ is infinite and $\sum_{|y_k|<\epsilon} |y_k| \cdot \nu_k$ is infinite for every $\epsilon > 0$ then – under additional assumptions – we can add an extra drift terms compensating growth of some of $y_k N^{(k)}(t)$ to get Lévy process. This is possible if we assume that

$$(a) \quad \sum_{|y_k|\geq\epsilon} \nu_k < \infty \qquad \text{and} \qquad (b) \quad \sum_{|y_k|<\epsilon} (y_k)^2 \, \nu_k < \infty \tag{26}$$

for some $\epsilon > 0$. Consider

$$\psi(p) = \sum_{k=1}^{\infty} \left(\mathrm{e}^{ipy_k} - 1 - ipy_k \mathbb{1}_{(-\epsilon,\epsilon)}(y_k) \right) \nu_k, \tag{27}$$

here by $\mathbb{1}_A(\cdot)$ we denote the indicator function of a set A. By (26) the series (27) is absolutely convergent uniformly in p on bounded intervals and function $\psi(\cdot)$ is a Lévy exponent corresponding to a Lévy process. We note that each distinct term $ip(-y_k)\nu_k$ in (27) corresponds to a deterministic process

$$C^{(k)}(t) := -y_k \nu_k \, t$$

which compensates the growth of $y_k N^{(k)}(t)$, as $\mathbb{E} y_k N^{(k)}(t) = y_k \nu_k \, t$. However, in general, a sum of infinitely many of these terms must be kept under the summation sign in (27) in order to make the series convergent.

Further generalization of Lévy process "built of" Poisson processes comes by replacing measure given by weights ν_k on discrete set of possible jump sizes $\{y_k : k \in \mathbb{N}\}$ with a more general measure $\nu(dy)$ supported on $\mathbb{R}\backslash\{0\}$ (see [3], p. xvii). The cases (ii), (iii) above generalize to the following (with $\epsilon = 1$).

(*ii*) The measure ν is not finite, but

$$\int_{\mathbb{R}\backslash\{0\}} (|y| \wedge 1) \, \nu(dy) < \infty, \tag{28}$$

here $a \wedge b = \min\{a, b\}$. In this case the function $y \to \mathrm{e}^{ipy} - 1$ is ν–integrable, and there exists a Lévy process with the Lévy exponent

$$\psi(p) = \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^{ipy} - 1) \nu(dy). \tag{29}$$

(iii) $\int_{(-1,1)\setminus\{0\}}(|y| \wedge 1)\,\nu(dy) = \infty$ but

$$\int_{\mathbb{R}\setminus\{0\}} (y^2 \wedge 1)\,\nu(dy) < \infty.$$

In this case there exist a Lévy process with the Lévy exponent

$$\psi(p) = \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{ipy} - 1 - ipy \cdot \mathbb{1}_{(-1,1)}(y)\right) \nu(dy). \tag{30}$$

Definition 13. *A Borel measure on* $\mathbb{R}\setminus\{0\}$ *satisfying*

$$\int_{\mathbb{R}\setminus\{0\}} (y^2 \wedge 1)\,\nu(dy) < \infty \tag{31}$$

is called Lévy measure.

It turns out that every Lévy process can be represented as a sum of the form

$$aX_1(t) + bX_2(t) + ct,$$

where a, b, c are some constants, $X_1(t)$ is a process with Lévy exponent (29) or (30), and $X_2(t)$ is a Brownian motion. For details see chapter 2.4 in [3] about the *Lévy–Itô decomposition*. The general formula for Lévy exponent is given by the Lévy–Khintchine theorem (see i.e. [3], p. 28, [18] p. 194).

Theorem 14 (Lévy–Khintchine)**.** *Characteristic function of every infinitely divisible random variable on* $\mathbb{R}$ *has the form* $\exp\{\psi(\cdot)\}$ *with the exponent*

$$\psi(p) = i\mathrm{m}p - \frac{1}{2}\sigma^2 p^2 + \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{ipy} - 1 - ipy \cdot \mathbb{1}_{(-1,1)}(y)\right) \nu(dy), \tag{32}$$

where $\mathrm{m} \in \mathbb{R}$, $\sigma^2 \geq 0$ *and* ν *is a Lévy measure. On the other hand, every function of this form is characteristic function of an infinitely divisible random variable.*

Remark 15. If the Lévy measure ν is finite, or satisfies $\int_{(-1,1)\setminus\{0\}} |y|\nu(dy) < \infty$, then the exponent (32) can be presented as

$$\psi(p) = i\widetilde{\mathrm{m}}p - \frac{1}{2}\sigma^2 p^2 + \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{ipy} - 1\right) \nu(dy),$$

where $\widetilde{\mathrm{m}} := \mathrm{m} - \int_{(-1,1)\setminus\{0\}} y\,\nu(dy)$.

We end this section with the following theorem.

Theorem 16 ([3], p. 31)**.** *Every infinitely divisible probability measure is a weak limit of compound Poisson measures.*

Proof. ([3], p. 31). By Theorem 10 it is enough to show pointwise convergence of characteristic functions. First, let us note that for nonzero complex number z we have

$$\lim_{n\to\infty} n(z^{1/n}-1) = \log z, \tag{33}$$

where $\log z$ is the principal value of complex logarithm of z. Now let X be infinitely divisible random variable with distribution μ, and let $p \mapsto \phi_X(p)$ be its characteristic function. For every $n \in \mathbb{N}$ there exist a measure μ_n satisfying $\mu = (\mu_n)^{(*n)}$, and $p \mapsto (\phi_X(p))^{1/n}$ is the characteristic function of μ_n. We define

$$\phi^{(n)}(p) := \exp\left\{n\left[\left(\phi_X(p)\right)^{1/n}-1\right]\right\}.$$

It follows by (33) that $\phi^{(n)}(p) \to \phi(p)$ as $n \to \infty$. Also, $\phi^{(n)}$ is the characteristic function of a compound Poisson, as

$$\phi^{(n)}(p) = \exp\left\{n\int\left(\mathrm{e}^{ipy}-1\right)\mu_n(dy)\right\}.$$

□

1.7 Stable laws

An important subclass of Lévy processes are *stable processes*. They consist of *stable random variables*. We recall that, by the classical central limit theorem, standard Gaussian random variable $X \sim \mathcal{N}(0,1)$ is a limit, in distribution, of any sequence of random variables $Z^{(n)}$, $n \in \mathbb{N}$ given by

$$Z^{(n)} = \frac{S^{(n)} - n\mathrm{m}}{\sigma\sqrt{n}}$$

if only $S^{(n)} = \sum_{k=1}^{n} Y^{(k)}$ are partial sums of a sequence of independent identically distributed random variables $Y^{(n)}$ with mean m and variance σ^2. Stable random variable generalizes this property of Gaussian.

Definition 17 ([3], p. 32)**.** *If a non–degenerate random variable X is the limit in distribution of $Z^{(n)}$ of the form*

$$Z^{(n)} = \frac{Y^{(1)} + Y^{(2)} + \cdots + Y^{(n)} - \mathrm{m}_n}{\sigma_n}$$

where $Y^{(n)}$, $n \in \mathbb{N}$ are i.i.d random variables and, for $n \in \mathbb{N}$, $\mathrm{m}_n \in \mathbb{R}$ and $\sigma_n > 0$, then X is called stable random variable.

The law of stable random variable is called *stable law*. By another equivalent definition, a non–degenerate random variable X is said to have a stable law if, given a sequence of independent identically distributed random variables $X^{(n)}, n \in \mathbb{N}$ with the same distribution as X, for every $n \in \mathbb{N}$ there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$a_n X + b_n \stackrel{\mathrm{d}}{=} X^{(1)} + X^{(2)} + \cdots + X^{(n)} \tag{34}$$

(see [22], p. 169, [18], p. 199). Here by $U \stackrel{\mathrm{d}}{=} V$ we understand that U and V have the same distribution. It turns out that the only possible values of a_n in (34) are $a_n = n^{1/\alpha}$ for some $0 < \alpha \leq 2$, and that $\alpha = 2$ iff $\mathbb{E}X^2 < \infty$ ([22], p. 170). It also holds that $\alpha \geq 1$ iff $\mathbb{E}|X| < \infty$ ([3], p. 33). Number α is called *index of stability*. Stable Lévy process with index α is called *α–stable*.

It immediately follows from (34) that X is infinitely divisible. The following two theorems characterize Lévy exponent of a stable random variable X.

Theorem 18 ([3] p. 33, [18] p. 200)**.** *Let X be a stable $\mathbb{R}$–valued random variable.*

– *If index of stability $\alpha = 2$, then X is a Gaussian random variable.*

– *If* $\alpha \in (0,2)$ *then the Lévy exponent of its characteristic function is*

$$\psi(p) = i\mathrm{m}p + \int_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{ipy} - 1 - ipy \cdot \mathbb{1}_{(-1,1)}(y)\right)\nu(dy),$$

with some $\mathrm{m} \in \mathbb{R}$ *and the Lévy measure* ν *being of the form*

$$\nu(dy) = \frac{c_+}{y^{1+\alpha}}\mathbb{1}_{(0,\infty)}(y)dy + \frac{c_-}{(-y)^{1+\alpha}}\mathbb{1}_{(-\infty,0)}(y)dy,$$

where constants c_+, c_- *are non-negative and at least one of them is positive.*

Theorem 19 ([18] p. 204)**.** *Let* X *be a stable* $\mathbb{R}$*–valued random variable with index of stability* $\alpha \in (0,2)$.

– *If* $\alpha \in (0,2)\setminus\{1\}$*, then*

$$\psi(p) = i\mathrm{m}p - \sigma^\alpha |p|^\alpha \left(1 + i\theta\,\mathrm{sgn}(p)\tan\frac{\pi}{2}\alpha\right)$$

with some $\mathrm{m} \in \mathbb{R}$*,* $\sigma > 0$ *and* $\theta \in [-1,1]$.

– *If* $\alpha = 1$*, then*

$$\psi(p) = i\mathrm{m}p - \sigma|p|\left(1 + i\theta\,\mathrm{sgn}(p)\frac{2}{\pi}\log|p|\right)$$

with m*,* σ *and* θ *as in the previous case.*

In the special case of symmetric stable random variable X with $\alpha \in (0,2]$ we have

$$\psi(p) = -\sigma^\alpha |p|^\alpha.$$

If X is stable with $\alpha < 2$ then it's tails decay polynomially, i.e. it holds ([3] p. 34)

$$\lim_{\lambda\to\infty} \lambda^\alpha \mathbb{P}[|X| > \lambda] = C > 0, \tag{35}$$

so the tails are "heavier" than Gaussian (which decay exponentially).

1.8 Markov processes and generators

A family of bounded linear operators $\{S_t : t \geq 0\}$ on a Banach space B is called a *semigroup*, if

(i) $S_t S_s = S_{t+s}$ for all $t, s \geq 0$, and

(ii) $S_0 = I$ (identity operator).

Denote norm of a Banach space B by $\|\cdot\|_B$. A semigroup $\{S_t : t \geq 0\}$ is called *strongly continuous*, if for every $f \in B$ the function $t \mapsto S_t f$ is continuous. If for all $t \geq 0$ and $f \in B$ we have $\|S_t f\|_B \leq \|f\|_B$, then $\{S_t : t \geq 0\}$ is called a *contraction semigroup*.

The *infinitesimal generator* of a strongly continuous semigroup $\{S_t : t \geq 0\}$ is the linear operator A defined as the limit (in the norm $\|\cdot\|_B$)

$$Af := \lim_{t \to 0^+} \frac{1}{t}\left(S_t f - f\right), \tag{36}$$

with domain $D(A)$ consisting of all $f \in B$ for which the limit exists.

Lemma 20. *The domain of infinitesimal generator of strongly continuous semigroup on a Banach space B forms a dense subset in B.*

Proof. (A sketch. For further details we refer to [21], Chapter 1.) It can be shown that for every $f \in B$ and $\epsilon > 0$ the Bochner integral $\int_0^\epsilon S_t f \, dt$ belongs to the domain of A. Furthermore, $\epsilon^{-1} \int_0^\epsilon S_t f \, dt$ has the limit f as $\epsilon \to 0^+$.

□

Let (E, d) be a metric space, i.e. a set E with a metric d on it. We denote the class of all Borel sets in E by $\mathcal{B}(E)$. Assume that $X = \{X(t) : t \geq 0\}$ is E–valued and adapted to a filtration $\{\mathcal{F}_t : t \geq 0\}$. We say that X is a *Markov process*, if

$$\mathbb{P}[X(t+s) \in A|\mathcal{F}_t] = \mathbb{P}[X(t+s) \in A|X(t)] \tag{37}$$

for every Borel set $A \subseteq E$ and for all $t, s \geq 0$. The property (37) is called *Markov property* of the process.

The law of $X(0)$ is called an *initial distribution* or *initial law* of the process.

A function $(t, x, A) \mapsto p(t, x, A) : [0, \infty) \times E \times \mathcal{B}(E) \to [0, 1]$ is called a *transition probability* function if

(i) $p(t, x, \cdot)$ is, for every $(t, x) \in [0, \infty) \times E$, a probabilistic measure on $\mathcal{B}(E)$,

(ii) $p(0, x, \cdot) = \delta_x(\cdot)$, which is Dirac measure at x, i.e. $\delta_x(A) = 1$ iff $x \in A$,

(iii) $p(\cdot, \cdot, A)$ is measurable and bounded on $[0, \infty) \times E$ for every $A \in \mathcal{B}(E)$,

(iv) the *Chapman–Kolmogorov property* holds: for arbitrary $t, s \geq 0$, $x \in E$ and $A \in \mathcal{B}(E)$ we have

$$p(t+s, x, A) = \int_E p(t, x, dy)p(s, y, A). \tag{38}$$

Theorem 21 (see [21], s. 157)**.** *Assume that (E, d) is a separable and complete metric space and $p(t, x, A)$ is a transition probability function. Let π be a probabilistic measure on the Borel sets $\mathcal{B}(E)$. Then there exists a Markov process*

$$X = \{X(t) : t \geq 0\},$$

whose finite dimensional distributions are given by

$$\begin{aligned} &\mathbb{P}[X(0) \in A_0, X(t_1) \in A_1, \dots, X(t_n) \in A_n] = \\ &\quad \int_{A_0} \cdots \int_{A_{n-1}} p(t_n - t_{n-1}, y_{n-1}, A_n) p(t_{n-1} - t_{n-2}, y_{n-2}, dy_{n-1}) \\ &\qquad \cdots p(t_1, y_0, dy_1)\pi(dy_0), \end{aligned}$$

where $t_1 < t_2 < \cdots < t_n$.

If the initial probability measure of X is δ_x for some $x \in E$, then we say that X is *starting at* x.

If a probability π satisfies

$$\pi(A) = \int_{\mathbb{E}} p(t, y, A)\pi(dy), \quad t > 0,\ A \in \mathcal{B}(E),$$

then π is called an *invariant* or *stationary* distribution of X. A Markov process X whose initial distribution is stationary is called *stationary*.

A discrete-time equivalent of the Markov process is called *Markov chain*. It is defined as a sequence of E–valued random variables $\xi_1, \xi_2 \dots$ such that for every set $A \in \mathcal{B}(E)$ we have

$$\mathbb{P}[\xi_{n+1} \in A | \xi_1, \dots, \xi_n] = \mathbb{P}[\xi_{n+1} \in A | \xi_n], \quad n \in \mathbb{N}.$$

Function $(x, A) \mapsto \mu(x, A)$, measurable in x for every $A \in \mathcal{B}(E)$ and being a probability measure on E for every x, is the transition probability function of a Markov chain $\{\xi_n : n \in \mathbb{N}\}$ if

$$\mathbb{P}[\xi_{n+1} \in A | \xi_n] = \mu(\xi_n, A), \quad n \in \mathbb{N},\ A \in \mathcal{B}(E).$$

An example of $\mathbb{R}$–valued Markov chain is

$$\xi_n := \sum_{i=1}^{n} Y^{(i)},$$

where $Y^{(n)}, n \in \mathbb{N}$ is sequence of independent identically distributed $\mathbb{R}$–valued random variables.

We come back to continuous-time Markov processes. A transition probability function $p(t, x, dy)$ gives rise to a semigroup of operators $\{P_t : t \geq 0\}$, on appropriate Banach space of functions on E, by the formula

$$P_t f(x) = \int_E f(y) p(t, x, dy). \tag{39}$$

The semigroup property $P_tP_s = P_{t+s}$ follows by the Chapman–Kolmogorov equation (38). The semigroup given by (39) on the space $B(E)$ of bounded measurable functions is referred as *Markov semigroup*. It is

(i) *positive*, which means that $f \geq 0 \Rightarrow P_t f \geq 0$ for all $f \in B(E)$, $t \geq 0$,

(ii) *conservative*, which means that $P_t\mathbb{1} = \mathbb{1}$ for $t \geq 0$,

(iii) a *contraction semigroup*, as it satisfies $\sup_x |P_t f(x)| \leq \sup_x |f(x)|$, $t \geq 0$.

A Markov process X with transition probability $p(t, x, dy)$ is related to the semigroup (39) by

$$\mathbb{E}[f(X(t+s))|\mathcal{F}_t] = \mathbb{E}[f(X(t+s))|X_t] = P_s f(X(t)). \tag{40}$$

In particular

$$P_t f(x) = \mathbb{E}_x f(X(t)),$$

where $\mathbb{E}_x$ is the expected value for the process starting at point x.

We now turn our attention to the characterization of Lévy processes by infinitesimal generators of their Markov semigroups. By $C_0(\mathbb{R})$ we denote the space of continuous functions on $\mathbb{R}$ vanishing at infinity, i.e. satisfying $\lim_{|x|\to\infty} f(x) = 0$, with the supremum norm.

Theorem 22 ([31] p. 118 and 119)**.** *Let $\{X(t) : t \geq 0\}$ be a Lévy process with the Lévy exponent*

$$\psi(p) = imp - \frac{1}{2}\sigma^2 p^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{ipy} - 1 - ipy \cdot \mathbb{1}_{(-1,1)}(y)\Big)\nu(dy). \tag{41}$$

Then the strongly continuous semigroup on $C_0(\mathbb{R})$ related to X by (40) *satisfies*

$$P_t f(x) = \int_{\mathbb{R}} f(x+y)\mu(t, dy), \qquad f \in C_0(\mathbb{R}), \tag{42}$$

where $\mu(t, dy)$ is the infinitely divisible measure whose Lévy exponent is $t\psi(\cdot)$. This semigroup has the infinitesimal generator given for each function f in the Schwartz space $\mathcal{S}(\mathbb{R})$ by the formula

$$\begin{aligned} Af(x) =& \mathrm{m}f'(x) + \frac{1}{2}\sigma^2 f''(x) \\ &+ \int_{\mathbb{R}\setminus\{0\}} \Big(f(x+y) - f(x) - yf'(x)\cdot \mathbb{1}_{(-1,1)}(y)\Big)\nu(dy). \end{aligned} \tag{43}$$

We see that the transition probability function $p(t, x, dy)$ of a Lévy process is the translation of $\mu(t, dy)$ by x:

$$p(t, x, A) = \mu(t, A - x). \tag{44}$$

In particular, if $\mu(t)$ has a density $\eta(t, y)$, then

$$P_t f(x) = [f * \eta(t)](x) = \int_{\mathbb{R}} f(y)\eta_t(x-y)dy.$$

We also note that the process X_x, defined as the Markov process starting at x with transition probability (44), is given by

$$X_x(t) = x + X(t), \qquad t \geq 0.$$

The translation-invariance property follows: the semigroup related to a Lévy process satisfies

$$P_t f(x+z) = P_t(\sigma_z f)(x),$$

where $\sigma_z f(x) := f(x+z)$ for every x, z. In probabilistic terms this means that

$$\mathbb{E}_{x+z} f(X(t)) = \mathbb{E}_x f(z + X(t)).$$

The formula (43) for the generator of Lévy process extends to $\mathbb{R}^d$–valued case in a straightforward manner, see i.e. [3] on page 139. In particular, the generator of d–dimensional standard Brownian motion is the Laplacian

$$Af(x) = \frac{1}{2}\Delta f(x) = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x),$$

the generator of Brownian motion with covariance $\{a_{ij}\}_{1\le i,j\le d}$ and drift $\{\mathrm{m}_i\}_{1\le i\le d}$ is given by the formula

$$Af(x) = \sum_{j=1}^{d} \mathrm{m}_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{45}$$

and the generator of Poisson process with intensity λ and jump by vector v is

$$Af(x) = \lambda[f(x+v) - f(x)].$$

At the end of this section we make one more note about general Markov process. An initial distribution π is called *reversible* measure, if for all $t \ge 0$

$$P(t, x, dy)\pi(dx) = P(t, y, dx)\pi(dy).$$

A Markov process $\{X_t : t \ge 0\}$ is called *reversible* if, for every $T > 0$, all finite dimensional distributions of processes $\{X_t, t \in [0, T]\}$ and $\{X_{T-t}, t \in [0, T]\}$ are identical.

Lemma 23 (see [54], p. 107-108)**.** *A process with transition semigroup $\{P_t\}$ is reversible iff its initial distribution is reversible.*

1.9 Some generalizations of Lévy processes

Compound Poisson process

Formula (43) in particular provides the generator of compound Poisson process with a finite Lévy measure ν

$$Af(x) = \int_{\mathbb{R}} (f(x+y) - f(x))\, \nu(dy). \tag{46}$$

We recall that the process generated by this operator has the representation

$$X(t) = \xi_{N(t)}, \quad \text{where} \quad \xi_n = \sum_{j=1}^{n} Y^{(j)}, \quad n = 1, 2, \ldots \tag{47}$$

here random variables $Y^{(j)}$, $j = 1, 2, \dots$, are independent identically distributed with common distribution $\nu(dy)/\nu(\mathbb{R})$. The values of variables $Y^{(1)}$, $Y^{(2)}, \dots$ are successive jumps. The times of jumps are chosen by Poisson process $N(t)$ with intensity equal to $\nu(\mathbb{R})$, independent of all $Y^{(j)}$. A more general compound Poisson process arises when we make the intensity vary depending on the actual position in the state space. We will describe such process in a more general state space, namely a metric space (E, d), which by assumption is separable and complete – such space is called a *Polish space*. Generator (46) generalizes as follows

$$Af(x) = \int_E (f(y) - f(x))\, R(x, dy). \tag{48}$$

We assume that for every $x \in E$ $R(x, \cdot)$ is a finite measure on $\mathcal{B}(E)$, and that the function $\lambda(x) := R(x, E)$ is bounded. Formula (46) indeed is a special case with $R(x, B) = \nu(B - x)$. Now we rewrite (48) in the following form

$$Af(x) = \lambda(x) \int_E (f(y) - f(x))\, \eta(x, dy),$$

where $\eta(x, dy) = R(x, dy)/\lambda(x)$. Then a compound Poisson process $\{X(t) : t \geq 0\}$ with the generator A is represented analogously to (47) as follows. Let $\{\zeta_n : n \in \mathbb{N}\}$ be the Markov chain on the state space E with transition probability $\eta(x, dy)$, and with the initial distribution (the distribution of ζ_1) being equal to the distribution of $X(0)$. Further let

$$t(n) := \sum_{i=1}^{n} \frac{\tau_i}{\lambda(\zeta_i)}$$

where $\tau_1, \tau_2, \dots$ are independent exponential random variables with parameter $\lambda = 1$. Then

$$X(t) = \zeta_{\widetilde{N}(t)}, \quad \text{where} \quad \widetilde{N}(t) := \max\{n : t(n) \leq t\}, \tag{49}$$

see [21] p. 162.

Diffusions

By letting drift an covariance coefficients in (45) vary, in appropriately regular manner, according to the position in space we obtain generators of path–continuous Markov processes called *diffusions*. Let us cite a theorem that gives sufficient conditions for transition probability function to generate such process. By $B_\epsilon(x)$ we denote open ball with radius ϵ centered at x,

$$B_\epsilon(x) := \{y \in \mathbb{R}^d : |y - x| < \epsilon\} \tag{50}$$

and by $B^c_\epsilon(x)$ we denote its complement.

Theorem 24 (see [56] on p. 252)**.** *Let functions* $\mathrm{m}_i(x)$ *and* $a_{ij}(x)$*, for* $i, j \in \{1, \ldots, d\}$*, be continuous on* $\mathbb{R}^d$*. Assyme that* $p(t, x, dy)$ *is a transition probability function on* $\mathbb{R}^d$ *such that for every* $\epsilon > 0$ *uniformly in* x *we have*

$$p\left(t, x, B^c_\epsilon(x)\right) = o(t),$$
$$\int\limits_{B_\epsilon(x)} (y_i - x_i)p(t, x, dy) = t \cdot \mathrm{m}_i(x) + o(t) \qquad \textit{and}$$
$$\int\limits_{B_\epsilon(x)} (y_i - x_i)(y_j - x_j)p(t, x, dy) = t \cdot a_{ij}(x) + o(t)$$

as $t \to 0$ *for* $i, j \in \{1, \ldots, d\}$*. Let*

$$P_t f(x) := \int_{\mathbb{R}^d} f(y)p(t, x, dy)$$

for any bounded and continuous function f *on* $\mathbb{R}^d$*. Then the limit*

$$Af := \lim_{t \to 0^+} t^{-1}(P_t f - f)$$

exists in supremum norm for every function f *being, together with all its first and second partial derivatives, bounded and continuous on* $\mathbb{R}^d$*, and for such functions it holds*

$$Af(x) = \sum_{j=1}^{d} \mathrm{m}_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

1.10 Itô integral

Let us denote by I an interval, which may be of the form $[0,\infty)$ or $[0,T]$ for some $T>0$. A stochastic process $\{X(t) : t \in I\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is *adapted* to a filtration $\{\mathcal{F}_t : t \in I\}$ if $X(t)$ is $\mathcal{F}_t$–measurable for every t. Process $\{X(t) : t \in I\}$ is called *measurable* if $X(t,\omega)$, as a function of $(t,\omega) \in I \times \Omega$, is measurable with respect to the σ–field $\mathcal{B}(I) \otimes \mathcal{F}$, where $\mathcal{B}(I)$ denotes the Borel σ–field of interval I.

A basic concept in stochastic calculus is the Itô stochastic integral. A thorough elaboration of the subject can be find i.e. in [29]. Below we survey the construction of the Itô integral with respect to the one–dimensional Brownian motion

$$w = \{w(t) : t \geq 0\}.$$

We assume that w is adapted to a filtration $\{\mathcal{F}_t : t \geq 0\}$, and $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for every $s < t$.

Definition 25 (Simple process)**.** *Let $T \in (0,\infty)$. A process $X = \{X(t) : t \in [0,T]\}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t : t \geq 0\}$ is called simple if for some $n \in \mathbb{N}$ it is of the form*

$$X(t,\omega) = X_0(\omega) \cdot \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{n-1} X_k(\omega) \cdot \mathbb{1}_{(t_k,t_{k+1}]}(t), \tag{51}$$

$$(t,\omega) \in [0,T] \times \Omega, \quad 0 = t_0 < t_1 < \cdots < t_n = T,$$

where for each k the random variable X_k is bounded and $\mathcal{F}_{t_k}$–measurable.

For a simple process (51) the stochastic integral over $[0,T]$ is defined by

$$\mathcal{I}_{[0,T]}(X) := \sum_{k=0}^{n-1} X_k \cdot \big(w(t_{k+1}) - w(t_k)\big), \tag{52}$$

and the stochastic integral over $[0,t]$, for any $t \in (0,T]$, is

$$\mathcal{I}_{[0,t]}(X) := \mathcal{I}_{[0,t]}\big(X|_{[0,t]}\big), \tag{53}$$

wherein by $X|_{[0,t]}$ we denote the restriction of X to the interval $[0,t]$ and the right hand side is given by (52) with n replaced by $\widetilde{n}$ such that $t_{\widetilde{n}-1} < t \leq t_{\widetilde{n}}$, and with $t_{\widetilde{n}}$ replaced by t. Here are some basic properties of (53).

Proposition 26. *Let X be simple on $[0,T]$. Then*

(i) if $X \equiv 1$ on $[0,T] \times \Omega$ then $\mathcal{I}_{[0,t]}(X) = w(t)$ for $t \in [0,T]$,

(ii) $\{\mathcal{I}_{[0,t]}(X) : t \in [0,T]\}$, considered with the filtration $\{\mathcal{F}_t : t \in [0,T]\}$, is a martingale,

(iii) if $0 < s \leq t \leq T$ then $\mathbb{E}\mathcal{I}_{[0,t]}(X)\mathcal{I}_{[0,s]}(X) = \mathbb{E}\int_0^s |X(u)|^2 du$.

For fixed $T > 0$ let $L_2^{\mathrm{A}}[0,T]$ be the space of all measurable and adapted stochastic processes X such that $\mathbb{E}\int_0^T |X(t)|^2 dt < \infty$. Surely simple processes on $[0,T]$ belong to $L_2^{\mathrm{A}}[0,T]$. Furthermore, the following proposition holds ([40] p. 45).

Lemma 27. *The space $L_2^{\mathrm{A}}[0,T]$ is a closed subspace of $L_2([0,T] \times \Omega)$ and for every X in $L_2^{\mathrm{A}}[0,T]$ there exists a sequence $X^{(n)}$, $n \in \mathbb{N}$ of simple processes on $[0,T]$ such that*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |X^{(n)}(t) - X(t)|^2 dt = 0. \tag{54}$$

According to point (3) of Proposition 26, a simple process X on $[0,T]$ satisfies

$$\left(\mathbb{E}\int_0^T (X(t))^2 dt\right)^{1/2} = \left(\mathbb{E}(\mathcal{I}_{[0,T]}(X))^2\right)^{1/2}. \tag{55}$$

It follows that $\mathcal{I}_{[0,T]}(\cdot)$ is an isometry of the space of simple processes and the space

$$\{\mathcal{I}_{[0,T]}(X) : X \text{ is simple on } [0,T]\},$$

considered with L_2–norms given respectively by the left and the right hand side of (55). By Lemma 27 simple processes form dense subset in $L_2^{\mathrm{A}}[0,T]$. The stochastic integral on $[0,T]$ for general $X \in L_2^{\mathrm{A}}[0,T]$ is defined as extension of the isometry to the whole of $L_2^{\mathrm{A}}[0,T]$:

Definition 28. *Let $X \in L_2^{\mathrm{A}}[0,T]$ and let $X^{(n)}$, $n \in \mathbb{N}$ be a sequence of simple processes for which* (54) *holds. A stochastic integral of X with respect to the Brownian motion is defined as the following limit in $L_2(\Omega)$:*

$$\mathcal{I}_{[0,T]}(X) := \lim_{n\to\infty} \mathcal{I}_{[0,T]}(X^{(n)}).$$

We denote it by

$$\int_0^T X(s)dw(s).$$

We stress here that $\mathcal{I}_{[0,T]}(X)$ is, strictly speaking, an equivalence class in $L_2(\Omega)$ of random variables. In particular, any two representatives of the integral are almost surely equal.

By $L_2^{\mathrm{A}}[0,\infty)$ we denote the space of all such processes $\{X(t) : t \geq 0\}$ that for every $T > 0$ the restriction $\{X(t) : t \in [0,T]\}$ is a member of $L_2^{\mathrm{A}}[0,T]$.

Proposition 29 (see [29], p. 137-140)**.** *The stochastic integral has following properties:*

(i) Let $X(\cdot), Y(\cdot) \in L_2^{\mathrm{A}}[0,\infty)$, and $\alpha, \beta \in \mathbb{R}$. Then

$$\int_0^t (\alpha X(s) + \beta Y(s))\, dw(s) = \alpha \int_0^t X(s)dw(s) + \beta \int_0^t Y(s)dw(s).$$

(ii) Let $X(\cdot) \in L_2^{\mathrm{A}}[0,\infty)$. The process

$$\left\{ \int_0^t X(s)dw(s) : t \geq 0 \right\}$$

is a square integrable martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

(iii) Let $X(\cdot), Y(\cdot) \in L_2^{\mathrm{A}}[0,\infty)$. Then

$$\mathbb{E} \int_0^t X(s)dw(s) \int_0^t Y(s)dw(s) = \int_0^t X(s)Y(s)ds.$$

Remark 30. Consider a partition $\Pi := \{t_0, t_1, \ldots, t_n\}$ of the interval $[0,t]$ where $0 = t_0 < \cdots < t_n = t$. Let $|\Pi| := \max\{t_1 - t_0, \ldots, t_n - t_{n-1}\}$ and $X(\cdot) \in L_2^{\mathrm{A}}[0,t]$. Assuming that $\mathbb{E}X(t)X(s)$ is continuous function in t and s we have (see [40], p. 57)

$$\int_0^t X(s)dw(s) := \lim_{|\Pi|\to 0} \sum_{k=0}^{n-1} X(t_k) \cdot \big(w(t_{k+1}) - w(t_k)\big),$$

here the limit is in $L_2(\Omega)$. In the approximating sum on the right hand side the value of $X(t_k)$ is multiplied by the increment of Brownian motion on interval $(t_k, t_{k+1}]$. Other approaches are possible leading to different integrals. The *Stratonovich integral* is important in applications. It is defined as

$$\int_0^t X(s) \circ dw(s) := \lim_{|\Pi|\to 0} \sum_{k=0}^{n-1} \frac{1}{2}\big(X(t_k) + X(t_{k+1})\big) \cdot \big(w(t_{k+1}) - w(t_k)\big).$$

A more general construction of Itô integral, which we do not present here, is made for an adapted integrand $X(t)$ satisfying the condition that $\int_0^T (X(s))^2 ds$ is finite almost surely, and does not necessarily belong to $L_2^{\mathrm{A}}[0,T]$. It can be found i.e. in [29] on page 146 or in Chapter 5 of [40].

Let us now express the celebrated Itô formula providing the differentiation rule for a process

$$\{f(w(t)) : t \geq 0\}$$

given that function $f(\cdot)$ is sufficiently smooth ([29], p. 149). By $C^2(\mathbb{R}^d)$ we denote the space of all functions which are continuous together with their first– and second–order derivatives.

Lemma 31 (Itô's Lemma)**.** *Let $f \in C^2(\mathbb{R})$ with f' bounded. Then almost surely*

$$f\big(w(t)\big) = f\big(w(0)\big) + \int_0^t f'\big(w(s)\big)dw(s) + \frac{1}{2}\int_0^t f''\big(w(s)\big)ds. \tag{56}$$

Differential form of the Itô formula reads

$$df\big(w(t)\big) = f'\big(w(s)\big)dw(s) + \frac{1}{2}\,f''\big(w(s)\big)ds. \tag{57}$$

Remark 32 ([30], p. 101)**.** The Stratonovich integral obeys the rules of classical calculus

$$f\big(w(t)\big) = f\big(w(0)\big) + \int_0^t f'\big(w(s)\big) \circ dw(s).$$

Below we state the multidimensional version of Itô formula (see i.e. [29], p. 153). It involves the notion of a stochastic integral with respect to a Brownian motion $W(t)$ with general variance $\mathbb{E}(W(t))^2 = at$, $a > 0$ (no drift). For a simple process $X(t)$ of the form (51) the integral is defined by

$$\mathcal{I}^W_{[0,T]}(X) := \sum_{k=0}^{n-1} X_k \cdot \big(W(t_{k+1}) - W(t_k)\big).$$

We have

$$\mathbb{E}(\mathcal{I}^W_{[0,T]}(X))^2 = a\,\mathbb{E}\int_0^T (X(t))^2 dt,$$

and if $\tilde{X} = \{\tilde{X}(t) : t \in [0,T]\}$ is a limit in $L_2^{\mathrm{A}}[0,T]$ of a sequence of simple processes $X^{(n)}$, then $\int_0^T \tilde{X}(t)dW(t)$ is defined as the limit of $\mathcal{I}^W_{[0,T]}(X^{(n)})$ in $L_2(\Omega)$.

Lemma 33 (Itô's lemma)**.** *Let* $W(t) = (W_1(t), \ldots, W_d(t))$ *be a d–dimensional Wiener process with covariance matrix* $\{a_{ij}\}_{1\le i,j\le d}$*, and let* $f(t,x) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ *be a continuous function having all derivatives* $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ *continuous with* $\frac{\partial f}{\partial x_i}$ *bounded. Then, for every* $t \geq 0$*,* $\mathbb{P}$ *almost surely it holds*

$$\begin{aligned} f(t, W(t)) = f(0, W(0)) &+ \int_0^t \frac{\partial}{\partial t} f(s, W(s))ds \\ &+ \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(s, W(s))dW_i(s) \\ &+ \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d a_{ij} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, W(s))ds. \end{aligned} \tag{58}$$

Remark 34. Assumptions on boundedness of the first-order derivatives of f in lemmas 31 and 33 are imposed only to ensure that stochastic integrals are given by definition 28. With a more general construction of Itô integral, which we do not present here, it suffices to assume that f belongs to $C^2(\mathbb{R})$, resp. $C^2(\mathbb{R}^d)$.

1.11 Stochastic differential equations

Let $\{w(t) : t \geq 0\}$, $w(t) = (w_1(t), \ldots, w_d(t))$ be a standard d–dimensional Brownian motion adapted to a filtration $\{\mathcal{F}_t : t \geq 0\}$, such that $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for $t > s$. Furthermore let

$$x \mapsto \sigma(x) = \{\sigma_{ij}(x)\}_{1\leq i\leq m, 1\leq j\leq d}$$

be a function of variable $x \in \mathbb{R}^m$ with values in space of $m \times d$ matrices, and let

$$x \mapsto b(x) = \{b_i(x)\}_{1\leq i\leq m}$$

be $\mathbb{R}^m$–valued function of $x \in \mathbb{R}^m$. We formulate a stochastic differential equation with initial condition ξ (see Chapter 12 in [56])

$$\begin{aligned} dX(t) &= b(X(t))dt + \sigma(X(t))dw(t), \\ X(0) &= \xi, \end{aligned} \tag{59}$$

which rewritten in coordinates reads

$$\begin{aligned} dX_i(t) &= b_i(X(t))dt + \sum_{j=1}^{d} \sigma_{ij}(X(t))dw_j(t), \\ X_i(0) &= \xi_i, \end{aligned}$$

for $i = 1, \ldots, m$. Solution of (59) is defined as a random process $\{X(t) : t \geq 0\}$ adapted to filtration $\{\mathcal{F}_t : t \geq 0\}$ and satisfying

$$X_i(t) = \xi_i + \int_0^t b_i(X(t))dt + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(X(t))dw_j(t) \tag{60}$$

for $t \geq 0$ and $i = 1, \ldots, m$. The following theorem holds, see [56] p. 286, [46] p. 66, [55] p. 124.

Theorem 35. *Assume that functions $\sigma_{i,j}(\cdot)$ and $b_i(\cdot)$ are Lipschitz on $\mathbb{R}^m$, i.e.*

$$|b_i(x) - b_i(y)| \leq D\|x - y\|, \quad \textit{and}$$
$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq D\|x - y\|$$

for some constant D and every $x, y \in \mathbb{R}^m$. Let ξ be a square integrable random variable independent of the Brownian motion and let $\widetilde{\mathcal{F}}_t$ be σ–field generated by ξ together with $\mathcal{F}_t$ for every $t \geq 0$. Then there exists a solution of (59) *adapted to filtration $\{\widetilde{\mathcal{F}}_t : t \geq 0\}$, which has almost surely continuous realizations and is bounded in norm of $L_2(\Omega)$ on every finite interval $[0, T]$. Furthermore, if $X(\cdot)$ and $X'(\cdot)$ are two solutions, then $\mathbb{P}[X(t) = X'(t)] = 1$ for every $t \geq 0$.*

Remark 36. In terms of Stratonovich differential equation (59) takes the following form ([30], p. 159)

$$dX_i(t) = [b_i(X(t)) - c_i(X(t))]dt + \sum_{j=1}^{d} \sigma_{ij}(X(t)) \circ dw_j(t),$$

$i = 1, \ldots, m$, with correction term given by the formula

$$c_i(x) = \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{d} \sigma_{jk}(x) \frac{\partial \sigma_{ik}}{\partial x_j}(x). \tag{61}$$

Let us outline a result about existence of solution to a stochastic differential equation in Hilbert space. First we note that given a pair of separable Hilbert spaces H and K, the space

$$L_2(H, K) := \{T : T \text{ is a Hilbert–Schmidt operator from } H \text{ into } K\}$$

with scalar product

$$\langle T_1, T_2\rangle := \sum_{n=1}^{\infty}\langle T_1 e_n, T_2 e_n\rangle_K,$$

where $\{e_n\}$ is any orthonormal basis in H, is a separable Hilbert space ([20], p. 418). Further, if $\{f_n : n \in \mathbb{N}\}$ is an orthonormal basis in H and $w_n(t)$, $n \in \mathbb{N}$ is a sequence of independent $\mathbb{R}$–valued standard Brownian motions then the series

$$W(t) = \sum_{n=1}^{\infty} f_n w_n(t), \qquad t \geq 0, \tag{62}$$

defines a *cylindrical* Wiener process. We assume that $W(t)$ is adapted to a filtration $\{\mathcal{F}_t : t \geq 0\}$ and that $W(t+s) - W(t)$ is independent of $\mathcal{F}_t$ for every t and s. $L_2(H,K)$–valued stochastic process $\Phi = \{\Phi(t) : t \in [0,T]\}$ is called simple if it satisfies

$$\Phi(t,\omega) = \Phi_0(\omega)\cdot \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{n-1}\Phi_k(\omega)\cdot \mathbb{1}_{(t_k,t_{k+1}]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, $\Phi_0, \ldots, \Phi_{n-1}$ are $L_2(H,K)$ valued random variables taking only finite number of values and Φ_k is $\mathcal{F}_{t_k}$–measurable for each k.

The stochastic integral of Φ over an interval $[0,T]$ with respect to $W(t)$ is defined by the formula

$$\int_0^T \Phi(s)dW(s) := \sum_{k=0}^{n-1}\Phi_k\big(W(t_{k+1}) - W(t_k)\big). \tag{63}$$

Note that by the above formula the integral is also defined over interval $[0,t]$ for every $t \in [0,T]$. We define norms $\|\Phi\|_t$, $t \in [0,T]$ by

$$\|\Phi\|_t^2 := \int_0^t \sum_{n\in\mathbb{N}} \|\Phi(s)f_n\|_K^2 \, ds,$$

and we have the isometry

$$\|\Phi\|_t^2 = \mathbb{E}\Big(\int_0^T \Phi(s)\, dW(s)\Big)^2,$$

which serves for the definition of stochastic integral with respect to $W(t)$, $t \in [0,T]$ of general $L_2(H,K)$–valued stochastic processes being limits of simple processes in norm $\|\cdot\|_T$. Now let us look at the following equation

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + G(X(t))dW(t), \\ X(0) = X_0 \end{cases} \tag{64}$$

where operator A with domain in H is an infinitesimal generator of a semigroup $\{S(t) : t \geq 0\}$, $F : H \to H$ is a measurable mapping, $G(x)$ is a Hilbert–Schmidt operator on H for every $x \in H$, and $G(\cdot)$ as $L_2(H) := L_2(H,H)$–valued mapping is measurable on H. Assume that the following Lipschitz conditions hold:

$$(i) \ \ \|F(x) - F(y)\|_H + \|G(x) - G(y)\|_{L_2} \leq C\|x - y\|_H, \quad x, y \in H,$$

$$(ii) \ \ \|F(x)\|_H^2 + \|G(x)\|_{L_2}^2 \leq C(1 + \|x\|_H^2), \quad x \in H,$$

for some $C > 0$, here $\|G\|_{L_2} = \left(\sum_{n=1}^{\infty} \|Ge_n\|_H^2\right)^{1/2}$ is the norm in $L_2(H)$. Let X_0 be $\mathcal{F}_0$–measurable H–valued random variable. Without entering here the details such as additional assumptions imposed on the filtration (see [20] on p. 75) and definition of a *mild* solution of stochastic differential equation (see the definition on page 182 of [20]), we note that Theorem 7.4 in [20], p. 186, establishes existence and uniqueness of process $X(t), t \in [0,T]$ solving (in a sense of mild solution) equation (64) given that (i) and (ii) hold. This solution satisfies

$$\int_0^T \|X(s)\|_H^2 ds < \infty \quad \text{almost surely,}$$

and has a continuous modification. For further details and more general statement we refer to [20].

Remark 37 (Diffusions as solutions of SDE)**.** Under the assumption that $\sigma_{ij}(\cdot)$ and $b_i(\cdot)$ are Lipschitz continuous, by Theorem 35 we have a path–continuous solution of

(59) with initial condition $X(0) = x$ for arbitrary $x \in \mathbb{R}^d$. Let us denote this solution by $X_x = \{X_x(t) : t \geq 0\}$. It follows that this is a Markov process with infinitesimal generator given by the formula

$$Af(x) = \sum_{j=1}^{m} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

for every continuous f with compact support and continuous partial derivatives of first and second order (see Theorem 4 on p. 298 in [56]). Coefficients a_{ij} are

$$a_{ij}(\cdot) = \sum_{k=1}^{d} \sigma_{ik}(\cdot)\sigma_{jk}(\cdot),$$

i.e. the matrix $a = \{a_{ij}\}$ is equal to $\sigma\sigma^T$.

1.12 Ornstein–Uhlenbeck process

The one–dimensional *Ornstein–Uhlenbeck process* (OU) is given as the solution of

$$d\xi(t) = \gamma\,(\mathrm{m} - \xi(t))dt + \sigma\sqrt{2\gamma}\,dw(t),$$

where γ and σ are positive numbers, and $\mathrm{m} \in \mathbb{R}$. This is a Gaussian, Markov process having a stationary measure. Note that the drift term $\gamma\,(\mathrm{m} - \xi(t))$ is proportional, with the opposite sign, to the deviation of the value $\xi(t)$ from m, so the process drifts towards the position m. The mean value of $\xi(t)$ approaches m as $t \to \infty$. This is called *mean–reverting* property. In particular, stationary distribution of OU process has mean m. From now on we assume that $\mathrm{m} = 0$ and consider

$$d\xi(t) = -\gamma\,\xi(t)dt + \sigma\sqrt{2\gamma}\,dw(t), \tag{65}$$

with initial condition $\xi(0) = \xi_0$ satisfying $\mathbb{E}\xi_0^2 < \infty$. The solution of (65) is given in explicit form ([29], p. 358)

$$\xi(t) = \mathrm{e}^{-\gamma t}\xi_0 + \sigma\sqrt{2\gamma}\int_0^t \mathrm{e}^{-\gamma(t-s)}dw(s).$$

The mean $\mathbb{E}\xi(t)$ is equal to $\mathrm{e}^{-\gamma t}\mathbb{E}\xi_0$, and the variance of $\xi(t)$ is

$$\mathrm{Var}(\xi(t)) = \sigma^2 + \left(\mathrm{Var}(\xi_0) - \sigma^2\right)\mathrm{e}^{-2\gamma t}. \tag{66}$$

In particular, the stationary measure π is Gaussian with mean 0 and variance σ^2. In the stationary case, i.e. if $\xi_0 \sim \mathcal{N}(0, \sigma^2)$, the process has covariance

$$\mathbb{E}\xi(t)\xi(s) = \sigma^2\,\mathrm{e}^{-\gamma|t-s|}.$$

There is an explicit formula for the transition probability function of OU defined by the equation (65). It reads (see i.e. [50], p. 21)

$$p(t, x, A) = \int \mathbb{1}_A\left(\mathrm{e}^{-\gamma t}\,x + \sqrt{1 - \mathrm{e}^{-2\gamma t}}\,y\right)\pi(dy), \tag{67}$$

here the integral is over $\mathbb{R}$.

Let us now focus on OU process in a much more general state space. Namely, let us consider a separable Banach space B and a centered Gaussian measure π on $\mathcal{B}(B)$ with reproducing kernel Hilbert space H embedded in B, so the triple (B, H, π) is an abstract Wiener space (as introduced in Section 1.3 on page 18). The formula (67) still defines a transition probability function if we let $x, y \in B$ and $A \in \mathcal{B}(B)$. We will show that the Chapman–Kolmogorov property holds. To this end, for brevity we denote $e_t := \mathrm{e}^{-\gamma t}$ and $h_t := \sqrt{1 - \mathrm{e}^{-2\gamma t}}$. By μ_t let us denote the following push–forward measure under mapping $x \mapsto J_{\sqrt{t}}x := \sqrt{t}\,x$ of π:

$$\mu_t(A) := \pi\{x : \sqrt{t}\,x \in A\}, \qquad A \in \mathcal{B}(B).$$

Then μ_{ts} is the push–forward measure of μ_s under $J_{\sqrt{t}}$:

$$\mu_{ts}(A) := \mu_s\{x : \sqrt{t}\,x \in A\}, \qquad A \in \mathcal{B}(B). \tag{68}$$

Let us denote the covariance operator of π by Q. It follows that μ_t is centered Gaussian with covariance operator tQ. We can represent the transition function $p(t, x, A)$ as follows:

$$\begin{aligned}p(t,x,A) &= \int_B \mathbb{1}_A\left(e_t x + h_t y\right)\pi(dy) = \int_B \mathbb{1}_{A-e_t x}\left(h_t y\right)\pi(dy)\\ &= \int_B \mathbb{1}_{A-e_t x}\left(y\right)\mu_{h_t^2}(dy) = \mu_{h_t^2}\left(A - e_t x\right),\end{aligned}$$

here $A - z := \{y - z : y \in A\}$. We note that $\mu_{t+s} = \mu_t * \mu_s$, and we calculate

$$\begin{aligned}\int_B p(t,y,A)p(s,x,dy) &= \int_B \mu_{h_t^2}\left(A - e_t y\right)\mu_{h_s^2}\left(dy - e_s x\right) \qquad (69)\\ &= \int_B \mu_{h_t^2}\left(A - e_t y - e_{t+s}x\right)\mu_{h_s^2}\left(dy\right)\\ &= \int_B \mu_{h_t^2}\left(A - e_{t+s}x - y\right)\mu_{e_t^2 h_s^2}\left(dy\right)\\ &= \int_B \mathbb{1}\left(A - e_{t+s}x\right)\mu_{h_t^2} * \mu_{e_t^2 h_s^2}\left(dy\right),\\ &= \mu_{h_{t+s}^2}\left(A - e_{t+s}x\right) = p(t+s,x,A).\end{aligned}$$

The Chapman–Kolmogorov property holds. The Markov process in B with transition probability $p(\cdot,\cdot,\cdot)$ is called Ornstein–Uhlenbeck process. Distribution π is stationary for the process as

$$\begin{aligned}\int_B p(t,x,A)\pi(dx) &= \int_B \mu_{h_t^2}\left(A - e_t x\right)\pi(dx) \qquad (70)\\ &= \int_B \mu_{h_t^2}\left(A - x\right)\mu_{e_t^2}(dx)\\ &= \mu_{h_t^2+e_t^2}(A) = \pi(A).\end{aligned}$$

Let us generalize Ornstein–Uhlenbeck process in yet another way. To this end we need some preliminaries. For $N \in \mathbb{N}$ we define sets

$$Z_N = \{-N+1, -N+2, \dots, N-1, N\}$$

and

$$T_N := Z_N/2N = \left\{\frac{m}{2N} : m \in Z_N\right\}.$$

We denote by $\mathbb{T}$ an interval $[-1/2, 1/2]$ with the topology of one–dimensional torus, i.e. with the endpoints $-1/2$ and $1/2$ identified. On $\mathbb{T}$ we define continuous functions $\gamma(\cdot)$ and $\sigma(\cdot)$ such that

$$\sigma(k) \geq 0, \quad \sigma(k) = \sigma(-k), \quad k \in \mathbb{T}, \quad \int_{\mathbb{T}} \mathbb{1}_{\{l:\sigma(l)=0\}}(k)dk = 0, \tag{71}$$

and $\gamma(\cdot)$ has the same properties, however with one stronger condition that $\gamma(k) > 0$ for all k. Given a finite sequence $x = \{x_z : z \in Z_N\}$, for an arbitrary continuous non-negative and even function F on $\mathbb{T}$ let $J_F^N x = \{(J_F^N x)_z : z \in \mathbb{Z}\}$ be defined as

$$(J_F^N x)_z := \sum_{y \in Z_N} x_y \sum_{k \in T_N} e^{2\pi i k(z-y)} \frac{F(k)}{2N}, \quad z \in \mathbb{Z}.$$

This is a $2N$–periodic sequence in variable z, and it is real valued given that F is even. Let $u \in l_1(\mathbb{Z})$. The discrete Fourier transform $\widehat{u}$, see the definition in section (1.1), is a continuous function on torus $\mathbb{T}$. Let $J_F^{N*} u = \{(J_F^{N*} u)_z : z \in \mathbb{Z}\}$ be defined as follows:

$$(J_F^{N*} u)_z := \begin{cases} \displaystyle\sum_{k \in T_N} e^{2\pi i k z}\, \widehat{u}(k) \frac{F(k)}{2N}, & z \in Z_N, \\ 0 & z \in \mathbb{Z} \backslash Z_N, \end{cases}$$

We have

$$(J_F^N x, u) = \sum_{z \in \mathbb{Z}} u_z \sum_{y \in Z_N} x_y \sum_{k \in T_N} e^{2\pi i k(z-y)} \frac{F(k)}{2N},$$

where $(x, u) = \sum_{z \in \mathbb{Z}} x_z u_z$. By substituting $-k$ for k we get

$$(J_F^N x, u) = (x, J_F^{N*} u). \tag{72}$$

Further let

$$\langle u, v \rangle_{F,N} := \sum_{k \in T_N} \widehat{u}(k) \widehat{v}(-k) \frac{F(k)}{2N}, \quad u, v \in l_2(\mathbb{Z}). \tag{73}$$

Given a sequence $X = \{X_z : z \in \mathbb{Z}\}$ of independent identically distributed Gaussian random variables with $X_0 \sim \mathcal{N}(0, 1)$ we define

$$\xi_{\sigma,z}^N := (J_{\sqrt{\sigma}}^N X)_z = \sum_{y \in Z_N} X_y \sum_{k \in T_N} e^{2\pi i k(z-y)} \frac{\sqrt{\sigma(k)}}{2N}. \tag{74}$$

It follows that

$$\xi_\sigma^N := \{\xi_{\sigma,z}^N : z \in \mathbb{Z}\}$$

is a $2N$–periodic sequence of Gaussian random variables. For $u \in l_1(\mathbb{Z})$ we have

$$(\xi_\sigma^N, u) = \sum_{z \in Z_N} X_z (J_{\sqrt{\sigma}}^{N*} u)_z,$$

so

$$\mathbb{E}(\xi_\sigma^N, u)(\xi_\sigma^N, v) = \sum_{y \in Z_N} (J_{\sqrt{\sigma}}^{N*} u)_y (J_{\sqrt{\sigma}}^{N*} v)_y =$$

$$= \langle u, v \rangle_{\sigma,N}.$$

The characteristic function of (ξ_σ^N, u) is

$$\mathbb{E}\mathrm{e}^{i(\xi_\sigma^N, u)} = \mathrm{e}^{-\frac{1}{2}\|u\|_{\sigma,N}^2}, \ u \in l_1(\mathbb{Z}),$$

where

$$\|u\|_{\sigma,N}^2 := \langle u, u \rangle_{\sigma,N}.$$

We will pass to the limit as $N \to \infty$. To this end we embed the space of all bounded sequences indexed by $\mathbb{Z}$ in an appropriate Hilbert space in which such limit exists. For a fixed sequence of positive numbers $\lambda = \{\lambda_z : z \in \mathbb{Z}\}$ such that

$$\sum_{z \in \mathbb{Z}} \lambda_z < \infty$$

we define a Hilbert space H_λ as

$$H_\lambda := \Big\{ (x_z)_{z \in \mathbb{Z}} : \sum_{z \in \mathbb{Z}} \lambda_z x_z^2 < \infty \Big\}$$

with the inner product

$$\langle x, x' \rangle_\lambda := \sum_{z \in \mathbb{Z}} \lambda_z x_z x_z',$$

and norm $\|x\|_\lambda^2 = \langle x, x \rangle_\lambda$. The dual of of H_λ is $H_{\lambda'}$, where $\lambda' := \{(\lambda_z)^{-1} : z \in \mathbb{Z}\}$. We note that $H_{\lambda'} \subset l_1(\mathbb{Z})$. Let us denote the distribution of ξ_σ^N on H_λ by μ_σ^N. Further, let continuous functions $\sigma_1(\cdot)$, $\sigma_2(\cdot)$ on $\mathbb{T}$ satisfy conditions (71). It holds

$$\mu_{\sigma_1}^N * \mu_{\sigma_2}^N = \mu_{\sigma_1+\sigma_2}^N \qquad \text{and}$$

$$\mu^N_{\sigma_1\sigma_2}(A) = \mu^N_{\sigma_1}\big(x : J^N_{\sqrt{\sigma_2}}x \in A\big), \quad A \in \mathcal{B}(H_\lambda).$$

Calculations analogous to (69) and (70) show that function

$$p^N(t,x,A) = \int \mathbb{1}_A\left(J^N_{e_t}x + J^N_{h_t}y\right)\mu^N_\sigma(dy), \tag{75}$$

where

$$e_t(k) := \mathrm{e}^{-\gamma(k)t}, \qquad h_t(k) := \sqrt{1-\mathrm{e}^{-2\gamma(k)t}},$$

has the Chapman–Kolmogorov property, and that the measure μ^N_σ is invariant for p^N. A Markov process $\{\xi^N_\sigma(t) : t \geq 0\}$ with transition probability function p^N and initial distribution μ^N_σ is stationary Gaussian with mean $\mathbb{E}\xi^N_\sigma(t) = 0$ and with covariance

$$\mathbb{E}(\xi^N_\sigma(t),u)(\xi^N_\sigma(s),v) = \frac{1}{2N}\sum_{k\in T_N} \mathrm{e}^{-\gamma(k)|t-s|}\widehat{u}(k)\widehat{v}(-k)\sigma(k), \tag{76}$$

for $u, v \in H_{\lambda'}$. In particular

$$\mathbb{E}\xi^N_{\sigma,y}(t)\xi^N_{\sigma,z}(s) = \frac{1}{2N}\sum_{k\in T_N} \mathrm{e}^{-\gamma(k)|t-s|}\mathrm{e}^{-2\pi ik(y-z)}\sigma(k),$$

so the random field $\{\xi^N_{\sigma,y}(t) : t \geq 0, y \in \mathbb{Z}\}$ is space and time stationary.

Now let us approach the limit. We say that the sequence of probability measures $\{\mu^N : N \in \mathbb{N}\}$ defined on a Borel σ–field of a topological space $(S, \mathcal{T})$ is weakly convergent to μ if

$$\lim_{N\to\infty}\int_S f(x)\mu^N(dx) = \int_S f(x)\mu(dx)$$

for every function $f : S \to \mathbb{R}$ which is bounded and continuous on $(S, \mathcal{T})$. We denote by $\widehat{\mu}$ the characteristic function of a measure μ defined on Borel sets of H_λ:

$$\widehat{\mu}(u) := \int_{H_\lambda} \mathrm{e}^{i(x,u)}\mu(dx), \quad u \in H_{\lambda'}.$$

Lemma 38. *As $N \to \infty$, the sequence of measures $\{\mu^N_\sigma : N \in \mathbb{N}\}$ is weakly convergent to a probability measure μ_σ on $\mathcal{B}(H_\lambda)$ having characteristic function*

$$\widehat{\mu}_\sigma(u) = \mathrm{e}^{-\frac{1}{2}\|u\|^2_\sigma} \quad \textit{where} \quad \|u\|^2_\sigma = \int_{\mathbb{T}} |\widehat{u}(k)|^2\sigma(k)dk.$$

Proof. According to the Lemma 2.1 on page 153 and Theorem 2.2 on page 154 in [47] we only need to check that the following conditions hold:

(i) $\lim_{N\to\infty} \widehat{\mu}_\sigma^N(u) = \widehat{\mu}_\sigma(u)$ for all $u \in H_{\lambda'}$;

(ii) $\lim_{n\to\infty} \sup_{N\in\mathbb{N}} \int_{H_\lambda} \sum_{|z|\geq n} \langle x, f_z\rangle_\lambda^2 \, \mu_\sigma^N(dx) = 0$, where $f_z := \delta_z/\sqrt{\lambda_z}$, $z \in \mathbb{Z}$, and

$$(\delta_z)_y = \begin{cases} 1, & z = y, \\ 0, & z \neq y. \end{cases}$$

We note that the vectors f_z, $z \in Z$ form a complete orthonormal system in H_λ.

The condition (ii) establishes conditional compactness of $\{\mu_\sigma^N : N \in \mathbb{N}\}$, while (i) ensures that every limit point must be μ_σ. First we show (i). We have

$$u \in H_{\lambda'} \subset l_1(\mathbb{Z}),$$

so $\widehat{u}$ is continuous on $\mathbb{T}$ and $\|\widehat{u}\|_{\sigma,N}^2$ converges to the integral $\int |\widehat{u}(k)|^2\sigma(k)dk = \|u\|_\sigma^2$. Now let us show the relative compactness of $\{\mu_F^N : N \in \mathbb{N}\}$. We estimate

$$\int_{H_\lambda} \langle x, f_z\rangle_\lambda^2 \, \mu_\sigma^N(dx) = \mathbb{E}(\xi_\sigma^N, \sqrt{\lambda_z}\delta_z)^2 = \lambda_z \|\delta_z\|_{\sigma,N}^2$$

$$= \lambda_z \sum_{k\in T_N} \frac{\sigma(k)}{2N} \leq \|\sigma\|_\infty \lambda_z,$$

so

$$\int_{H_\lambda} \sum_{|z|\geq n} \langle x, f_z\rangle_\lambda^2 \, \mu_\sigma^N(dx) \leq \|\sigma\|_\infty \sum_{|z|\geq n} \lambda_z,$$

and (ii) follows. □

Let

$$\mathrm{R}(u, v) := \int_{\mathbb{T}} \widehat{u}(k)\widehat{v}(-k)\sigma(k)dk, \quad u, v \in H_{\lambda'}.$$

Function $(u, v) \mapsto \mathrm{R}(u, v)$ is positive definite on $H_{\lambda'} \times H_{\lambda'}$. We consider a Gaussian random field $\{\xi_\sigma(u) : u \in H_{\lambda'}\}$ with mean zero and covariance

$$\mathbb{E}\xi_\sigma(u)\xi_\sigma(v) = \mathrm{R}(u, v).$$

Such a random field may be constructed as the Gaussian random field of variables

$$\xi_\sigma(u) := (\cdot, u), \quad u \in H_{\lambda^*}$$

on probability space $(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$ with measure μ_σ established by Lemma 38. In particular, the sequence of random variables $\{\xi_\sigma(\delta_z) : z \in \mathbb{Z}\}$ is represented by the identity mapping $x \mapsto x,\ x \in H_\lambda$ on probability space $(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$, since $x = \{x_z : z \in \mathbb{Z}\}$ has coordinates $x_z = (x, \delta_z)$. We denote $\xi_{\sigma,z} := \xi_\sigma(\delta_z)$ and $\xi_\sigma := \{\xi_{\sigma,z} : z \in \mathbb{Z}\}$, so ξ_σ is a H_λ–valued random variable and we can identify $\xi_\sigma(u)$ with (ξ_σ, u) for $u \in H_{\lambda'}$. Now let us consider a function

$$\widetilde{R}\big((y,t),(z,s)\big) := \int_{\mathbb{T}} \mathrm{e}^{-2\pi i k(y-z)} \mathrm{e}^{-\gamma(k)|t-s|} \sigma(k) dk, \quad y, z \in \mathbb{Z},\ t, s \geq 0.$$

This function is positive definite on $T \times T$, where $T := \{(z,t) : z \in \mathbb{Z}, t \geq 0\}$. Indeed, $\widetilde{R}$ is the limit, as $N \to \infty$, of (76) with $u = \delta_y$ and $v = \delta_z$, which in turn is positive definite as a covariance of a Gaussian field. Now let us define a space–time stationary Gaussian random field $\{\xi_{\sigma,z}(t) : z \in \mathbb{Z}, t \geq 0\}$ with mean zero and with covariance

$$\mathbb{E}\xi_{\sigma,y}(t)\xi_{\sigma,z}(s) = R\big((y,t),(z,s)\big).$$

As we have already established, for fixed t we have that $\xi_\sigma(t) := \{\xi_{\sigma,z}(t) : z \in \mathbb{Z}\}$ is a H_λ–valued random variable with distribution μ_σ. It follows that $\{\xi_\sigma(t) : t \geq 0\}$ is a stationary Gaussian H_λ–valued stochastic process.

Remark 39. We do not have transition probabilities for $\{\xi_\sigma(t) : t \geq 0\}$. We have constructed it only as a stationary Gaussian random field.

1.13 Gaussian Hilbert space and stochastic integral – an example

By definition, a *Gaussian Hilbert space* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a closed subspace of the Hilbert space $L_2(\Omega, \mathcal{F}, \mathbb{P})$ consisting of centered Gaussian random variables (see [27], p. 4).

Remark 40. Every sequence of Gaussian random variables convergent in L_2 has limit being Gaussian random variable. It follows that the closure in L_2–norm of arbitrary linear space consisting of Gaussian random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian Hilbert space ([27], p. 4).

Definition 41 ([27], p. 9)**.** *Assume that $(E, \mathcal{A}, \pi)$ is a measure space, that is, $\mathcal{A}$ is a σ–algebra of subsets of E and π is a measure on $\mathcal{A}$. A Gaussian stochastic integral on $(E, \mathcal{A}, \pi)$ is a linear isometry of $L_2(E, \mathcal{A}, \pi)$ into a Gaussian Hilbert space.*

Let us consider Gaussian field $\xi_\sigma = \{\xi_{\sigma,z} : z \in \mathbb{Z}\}$ from the previous section – with values in H_λ and with covariance $\mathbb{E}(\xi_\sigma, u)(\xi_\sigma, v) = \int_{\mathbb{T}} \widehat{u}(k)\widehat{v}(-k)\sigma(k)dk$. The law of ξ_σ is denoted by μ_σ. The collection

$$\mathcal{H}_1 := \{(\cdot, u) : u \in H_{\lambda'}\}$$

is a linear space of Gaussian random variables on $(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$. We denote by $L_2(\sigma)$ the Hilbert space of functions on torus $\mathbb{T}$ with finite norm

$$\|f\|_{L_2(\sigma)} = \left(\int_{\mathbb{T}} |g(k)|^2 \sigma(k)dk\right)^{1/2}.$$

Let

$$\mathcal{G} := \{\widehat{u} : u \in H_{\lambda'}\}.$$

We consider $\mathcal{G}$ with the norm $\|\cdot\|_{L_2(\sigma)}$ and $\mathcal{H}_1$ with the norm of $L_2(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$. The mapping $\mathcal{I} : \mathcal{G} \mapsto \mathcal{H}_1$, $\mathcal{I}\widehat{u} = (\cdot, u)$ is an isometry. $L_2(\sigma)$–closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ is a Hilbert subspace consisting of all such $g \in L_2(\sigma)$ that

$$g(-k) = g^*(k), \quad k \in \mathbb{T}.$$

Note that with $\mathrm{e}_z(k) := \mathrm{e}^{-2\pi ikz}$, for $z \in \mathbb{Z}$ and $k \in \mathbb{T}$

$$\mathcal{I}\Big(\sum_{j=1}^{n} a_j \mathrm{e}_{z_j}\Big) = \sum_{j=1}^{n} a_j \xi_{\sigma,z_j} \tag{77}$$

for arbitrary $n \in \mathbb{N}$, arbitrary real numbers $a_1, \dots, a_n$ and integers $z_1, \dots, z_n$. We will construct a *stochastic spectral measure* (see [27], p. 112) corresponding to the random field $\{\xi_{\sigma,z} : z \in \mathbb{Z}\}$. To this end we extend the isometry to the whole space $L_2(\sigma)$ (we will denote the extension also by $\mathcal{I}$) in the following way. We admit complex coefficients a_j in (77), and we take limits, in L_2–norm, of trigonometric polynomials of the form $\sum_{j=1}^{n} a_j \mathrm{e}_{z_j}$, and define the isometric image of such limit as the limit of the corresponding linear combinations of Gaussians on the right hand side of (77). Since trigonometric polynomials with complex coefficients form a dense set in $L_2(\sigma)$, the isometry is then defined on the whole of $L_2(\sigma)$. We allow complex coefficients a_j, hence we deal with *complex Gaussian random variables* on the right hand side of (77). By definition, a complex random variable ζ is Gaussian, if $\mathrm{R}e\zeta$ and $\mathrm{I}m\zeta$ are jointly Gaussian ([27], p. 12). The isometric image $\mathcal{I}\big(L_2(\sigma)\big)$ is a Hilbert space consisting of complex Gaussian random variables. We have $\mathbb{E}\mathcal{I}g(\mathcal{I}h)^* = \langle f, g\rangle_{L_2(\sigma)}$.

If $A \in \mathcal{B}(\mathbb{T})$, then $\mathcal{I}\mathbb{1}_A$ has variance $\sigma(A) := \langle \mathbb{1}_A, \mathbb{1}_A\rangle_{L_2(\sigma)} = \int_A \sigma(k)dk$. We define random function $\widehat{\mu}_\sigma$ on $\mathcal{B}(\mathbb{T})$ by

$$\widehat{\mu}_\sigma(A) := \mathcal{I}\mathbb{1}_A.$$

If sets $A_1, \dots, A_n$ are disjoint then

$$\widehat{\mu}_\sigma\left(A_1 \cup \dots \cup A_n\right) = \sum_{i=1}^{n} \widehat{\mu}_\sigma(A_i).$$

Now consider a simple function

$$g(k) = \sum_{i=1}^{n} a_i \, \mathbb{1}_{A_i}(k),$$

where $A_1, \dots, A_n$ are disjoint. For such function we have

$$\mathcal{I}g = \sum_{i=1}^{n} a_i \mathcal{I}\mathbb{1}_{A_i} = \sum_{i=1}^{n} a_i\, \widehat{\mu}_\sigma(A_i),$$

and if a sequence of simple functions g_n is convergent to some $g_0 \in \overline{\mathcal{G}}$ in $L_2(\sigma)$, then $\mathcal{I}g_n$ converges to $\mathcal{I}g_0$ in a sense that the real (resp. imaginary) part of $\mathcal{I}g_n$ converges to the real (imaginary) part of $\mathcal{I}g$ in $L_2(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$. For $g \in L_2(\sigma)$, we call $\mathcal{I}g$ a *stochastic integral* of g with respect to the spectral measure $\widehat{\mu}_\sigma$ and we denote this integral by

$$\int_{\mathbb{T}} g(k)\, \widehat{\mu}_\sigma(dk).$$

We write formally that $\widehat{\mu}_\sigma(dk) = \widehat{\xi}_\sigma(dk) = \sum_{z\in\mathbb{Z}} \xi_{\sigma,z} \mathrm{e}^{-2\pi i k z}\, dk.$

Consider now the following families of random variables

$$\mathcal{P}_n := \big\{ f\big((\cdot, u_1), \dots, (\cdot, u_m)\big) : u_1, \dots, u_m \in H_{\lambda'},$$
$$\text{and } f \text{ is a polynomial of degree at most } n. \big\}$$

A Gaussian random variable X has all moments $\mathbb{E}|X|^p$, $p \in \mathbb{N}$ finite, and by using the Cauchy–Schwarz inequality we conclude that $\mathcal{P}_n$ are linear subspaces of $L_2(\mu_\sigma)$. Denote by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in $L_2(\mu_\sigma)$, and let

$$H^{:n:} := \overline{\mathcal{P}}_n \cap \overline{\mathcal{P}}_{n-1}^{\perp}, \quad n \geq 1.$$

Also let $H^{:0:} := \overline{\mathcal{P}}_0$, which is the space of constant functions. Here, for a subspace $\mathcal{E} \subset L_2(\mu_\sigma)$, $\mathcal{E}^\perp$ stands for the orthogonal complement of $\mathcal{E}$ in $L_2(\mu_\sigma)$. In particular, $H^{:1:} = \overline{\mathcal{H}}_1$. It follows that $H^{:n:}$, $n \geq 0$ is a sequence of orthogonal, closed subspaces of $L_2(\mu_\sigma)$ of functions measurable with respect to σ–field $\mathcal{F}$ generated by all Gaussian random variables in $\overline{\mathcal{H}}_1$, and $\overline{\mathcal{P}}_n$ is equal to the direct sum

$$\overline{\mathcal{P}}_n = H^{:0:} \oplus \dots \oplus H^{:n:}.$$

The space $L_2(H_\lambda, \mathcal{F}, \mu_\sigma)$ has the following *Wiener chaos decomposition* (see Theorem 2.6 in [27])

$$L_2(H_\lambda, \mathcal{F}, \mu_\sigma) = \bigoplus_{n=0}^{\infty} H^{:n:},$$

which means that $X \in L_2(H_\lambda, \mathcal{F}, \mu_\sigma)$ can be represented as

$$X = \sum_{n=1}^{\infty} \mathrm{P}_n X, \quad \text{where } \mathrm{P}_n \text{ is the orthogonal projection onto } H^{:n:} \text{ for each } n.$$

There is a Markov semigroup $\{P(t) : t \geq 0\}$ on $L_2(H_\lambda, \mathcal{F}, \mu_\sigma)$ related to the Ornstein–Uhlenbeck process $\xi_\sigma(t)$, constructed with use of chaos decomposition. For a general information about the Ornstein–Uhlenbeck semigroup on a Gaussian Hilbert space we refer to Chapter 4 of [27]. This semigroup preserves closed subspaces $H^{:n:}$. In particular, for any constant $C \in H^{:0:}$ it is defined as $P(t)C = C$. On elements of $H^{:1:} = \overline{\mathcal{H}}_1$ the semigroup acts as follows. For $u \in H_\lambda$ let us denote by $X_{\widehat{u}}$ the Gaussian $X_{\widehat{u}}(\cdot) := (\cdot, u)$, and let us extend this notation to every element of $\overline{\mathcal{H}}_1$, so $X_g = \mathcal{I}g$, $g \in \overline{\mathcal{G}}$. Then

$$P(t)X_g = X_{S(t)g}, \qquad S(t)g(k) := \mathrm{e}^{-\gamma(k)t} g(k).$$

Let us denote the generator of $P(t)$ by $\mathfrak{Q}$. It holds

$$\mathfrak{Q}X_g = X_{Qg}, \quad \text{where} \quad Qg(k) = -\gamma(k)g(k).$$

This may be concluded by noting that Q is the generator of $S(t)$ on $\overline{\mathcal{G}}$, using linearity $aX_g + bX_h = X_{ag+bh}$ and isometry. We note that the generator $\mathfrak{Q}$ is, by assumptions on γ, bounded on $\overline{\mathcal{G}}$

$$\mathbb{E}_{\mu_\sigma}(X_{Qg})^2 = \int_{\mathbb{T}} |\gamma(k)g(k)|^2 \sigma(k)\, dk \leq (\min_k \gamma(k))^2 \|g\|_\sigma^2,$$

and hence it is defined on whole of $\overline{\mathcal{G}}$. On $H^{:n:}$, $n \geq 2$, $P(t)$ is defined as follows. Given $X_1, \ldots X_n \in H^{:1:}$, we take $\mathrm{P}_n \prod_{k=1}^{n} X_k \in H^{:n:}$ and we set

$$P(t)\, \mathrm{P}_n \prod_{k=1}^{n} X_k := \mathrm{P}_n \prod_{k=1}^{n} P(t) X_k.$$

Such operator extends in a unique way to a bounded linear operator on $H^{:n:}$ (we refer to [27], Theorem 4.5). Eventually, for arbitrary X in $L_2(H_\lambda, \mathcal{F}, \mu_\sigma)$

$$P(t)X := \sum_{n=0}^{\infty} P(t)\mathrm{P}_n X.$$

Chapter 2

Model of energy transport at different scales

2.1 Stochastic dynamics at the microscale

We consider a chain of harmonic oscillators indexed by $y \in \mathbb{Z}$ described in [10, 11, 13]. We recall that energy of oscillator at y is given by

$$\mathfrak{e}_y := \frac{1}{2}\mathfrak{p}_y^2 + W(\mathfrak{q}_y) + \frac{1}{2}\sum_{|y-y'|=1} V(\mathfrak{q}_y - \mathfrak{q}_{y'}),$$

and here

$$W(\mathfrak{q}_y) + \frac{1}{2}\sum_{|y-y'|=1} V(\mathfrak{q}_y - \mathfrak{q}_{y'})$$

is the potential energy. If potentials $W(\cdot)$ and $V(\cdot)$ are quadratic, and interactions between oscillators is not restricted to the nearest neighbors, then the sum over all $y \in \mathbb{Z}$ of potential energies generalizes to the form

$$b(0)\sum_{y\in\mathbb{Z}} \mathfrak{q}_y^2 + \sum_{y,z\in\mathbb{Z}} b(z)(\mathfrak{q}_y - \mathfrak{q}_{y+z})^2, \tag{78}$$

where $z \mapsto b(z) \in \mathbb{R}$ is a nonnegative, even function on $\mathbb{Z}$. With

$$\alpha(y) := \begin{cases} b(0) + 2\sum_{|z|>0} b(z) & \text{if } y = 0, \\ -2b(y) & \text{if } y \neq 0 \end{cases}$$

the Hamiltonian rewrites as

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) = \frac{1}{2}\sum_{y\in\mathbb{Z}} \mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}} \alpha(y-y')\mathfrak{q}_y\mathfrak{q}_{y'}. \tag{79}$$

The discrete Fourier transform $\widehat{\alpha}(k)$ of potential $\alpha = \{\alpha(y) : y \in \mathbb{Z}\}$ is real and even. We have $\alpha(0) \geq \left|\sum_{|z|>0} \alpha(z)\right|$, and it follows that $\widehat{\alpha}$ is also nonnegative. A function

$$\omega(k) := \sqrt{\widehat{\alpha}(k)}, \quad k \in \mathbb{T},$$

is called *dispersion relation* of the chain. Note that $\widehat{\alpha}(0)$ is equal to

$$\sum_{y\in\mathbb{Z}} \alpha(y) = b(0) + 2\sum_{|z|\geq 1} b(z) - 2\sum_{|y|\geq 1} b(y) = b(0),$$

so the pinning potential W is present if and only if $\omega(0) > 0$.

The stochastic dynamics (3) turns into the following wave equation

$$\begin{cases} d\mathfrak{q}_y(t) = \mathfrak{p}_y(t)dt \\ d\mathfrak{p}_y(t) = -(\alpha * \mathfrak{q}(t))_y dt + d\xi_y^{(\epsilon)}[\mathfrak{p}](t), \end{cases} \tag{80}$$

where $(\alpha * \mathfrak{q})_y$ is the discrete convolution

$$(\alpha * \mathfrak{q})_y = \sum_{z\in\mathbb{Z}} \alpha(z)\mathfrak{q}_{y-z}.$$

2.2 The wave function and the Wigner transform

We refer to the following definition of the *wave function* $\psi = \{\psi_y : y \in \mathbb{Z}\}$ of the chain:

$$\psi_y := (\widetilde{\omega} * \mathfrak{q})_y + i\mathfrak{p}_y, \quad y \in \mathbb{Z}. \tag{81}$$

Here $\widetilde{\omega}$ is the inverse Fourier transform of the dispersion relation ω. In sightly different definition in [13], the right hand side of (81) is multiplied by number $2^{-1/2}$. In terms of wave numbers k the wave function is represented as the Fourier transform of (81)

$$\widehat{\psi}(k) = \omega(k)\widehat{\mathfrak{q}}(k) + i\widehat{\mathfrak{p}}(k), \quad k \in \mathbb{T}.$$

In the regime of appropriate time and space scalings, scaled $\frac{1}{2}|\psi_y|^2$ approximates the energy $\mathfrak{e}_y$ of the oscillator at position y, and $\frac{1}{2}|\widehat{\psi}(k)|^2$ is interpreted as density of energy carried by the component of vibration of the lattice with the wave number k.

In the purely deterministic harmonic system the dynamics of the wave function in the domain of k would be

$$\frac{d}{dt}\widehat{\psi}(t,k) = i\,\omega(k)\widehat{\psi}(t,k). \tag{82}$$

For $\psi_0 := \{\psi_{0,y} : y \in \mathbb{Z}\}$, a solution of (82) with initial condition $\widehat{\psi}(0,k) := \widehat{\psi}_0(k)$ is given by the formula

$$\widehat{\psi}(t,k) = \mathrm{e}^{i\omega(k)t}\widehat{\psi}_0(k).$$

For a moment let us fix $k \in \mathbb{T}$ and consider

$$\psi_{0,y} := e^{-2\pi iky}, \qquad y \in \mathbb{Z}.$$

This function surely does not belong to $l_2(\mathbb{Z})$. Its Fourier transform is Dirac measure on torus given by $\widehat{\psi}_0(dk') = \delta(k+k')dk'$ and

$$\psi_y(t) = \int_{\mathbb{T}} \mathrm{e}^{i\omega(k)t+2\pi ik'y}\delta(k+k')dk' = \mathrm{e}^{-2\pi ik\left(y-\frac{\omega(k)}{2\pi k}t\right)}, \quad y \in \mathbb{Z},$$

so the wave ψ_0 with the wave number k travels at a constant velocity $v = \omega(k)/(2\pi k)$ ([24], p. 47). Let us now consider the following initial condition

$$\psi_{0,y} := \int_{\mathbb{T}} e^{-2\pi iky}\,\widehat{\psi}_0(k)dk, \qquad y \in \mathbb{Z},$$

where $\widehat{\psi}_0 \in L_2(\mathbb{T})$. This time the solution is given by

$$\psi_y(t) = \int_{\mathbb{T}} e^{-2\pi ik\left(y-\frac{\omega(k)}{2\pi k}t\right)}\,\widehat{\psi}_0(k)dk, \quad y \in \mathbb{Z}$$

(we recall that $\omega(\cdot)$ is even). A derivation of the *group velocity* of a *wave packet* can be found in [24] on page 47. For a function $\widehat{\psi}_0(\cdot)$ which has a narrow peak at some $k_0 \in \mathbb{T}$, and is negligible outside a small vicinity of k_0, by expanding ω into Taylor series at k_0 and ignoring powers of k bigger than 1 we get

$$|\psi_y(t)| \sim \Big| \int_{\mathbb{T}} e^{-ik\left(2\pi y - \omega'(k_0)\cdot t\right)} \, \widehat{\psi}_0(k)\, dk \Big|.$$

Roughly speaking, the modulus $|\psi_y(t)|$ travels at velocity $\omega'(\cdot)/(2\pi)$ through the body.

Remark 42. Note that for the solution of (82) with initial condition $\widehat{\psi}(0,\cdot)$ in $L_2(\mathbb{T})$, $\int_{\mathbb{T}} |\psi(t,k)|^2 dk$ does not change in time t. It follows that it is so in perturbed system, as the perturbation is postulated to preserve total energy of the oscillators. However in the deterministic case we can additionally say that $|\psi(t,k)|^2$ is preserved for each fixed k, and in perturbed system this is not true.

Remark 43. Looking at the above deterministic equation we observe the following fact. As we scale the time variable by setting

$$\widehat{\psi}^{(\epsilon)}(t,k) := \widehat{\psi}(t/\epsilon, k),$$

we get equation

$$\frac{d}{dt}\widehat{\psi}^{(\epsilon)}(t,k) = \frac{i}{\epsilon}\,\omega(k)\widehat{\psi}(t,k),$$

which has the formal solution

$$\widehat{\psi}^{(\epsilon)}(t,k) = \mathrm{e}^{i\omega(k)t/\epsilon}\widehat{\psi}^{(\epsilon)}(0,k).$$

This solution becomes fast oscillating as $\epsilon \to 0$.

The object of main importance that stores information about energy distribution in the chain is the *Wigner transform* of the wave function. The dynamics is stochastic, and the averaged Wigner transform is considered, which is formally given by the formula

$$W(t,x,k) := \frac{1}{2}\,\mathbb{E}\int_{\mathbb{R}} \mathrm{e}^{2\pi i p x}\, \widehat{\psi}\left(t, k + \frac{p}{2}\right) \widehat{\psi}^* \left(t, k - \frac{p}{2}\right) dp. \qquad (83)$$

Here $\widehat{\psi}^*$ is the complex conjugation of $\widehat{\psi}$, and te mean value $\mathbb{E}$ is taken with respect to the initial measure and realization of the stochastic perturbation. The following relation holds

$$\int_{\mathbb{R}} W(t,x,k)dx = \frac{1}{2}\mathbb{E}|\widehat{\psi}(t,k)|^2. \tag{84}$$

On the other hand, if we denote by $\psi_{\epsilon,y}$ ($y \in \mathbb{Z}$), and by $W_\epsilon(t,x,k)$, the wave function and the Wigner transform of appropriately scaled model with scaling parameter ϵ, converging to a macroscopic dynamics as $\epsilon \to 0$, then $\int_{\mathbb{T}} W_\epsilon(t,x,k)dk$, $x \in \mathbb{R}$ asymptotically describes the energy distribution given by $\frac{1}{2}\mathbb{E}|\psi_{\epsilon,x}|^2$ concentrated on atoms $x \in \mathbb{Z}$ (embedded in $\mathbb{R}$).

The evolution of the Wigner transform is examined along with the *anti–Wigner transform*, which is formally defined as

$$Y(x,k) := \frac{1}{2}\,\mathbb{E}\int_{\mathbb{R}} \mathrm{e}^{2\pi i p x}\,\widehat{\psi}\left(-k+\frac{p}{2}\right)\widehat{\psi}\left(k+\frac{p}{2}\right)dp, \tag{85}$$

because $W(t,x,k)$ alone does not satisfy closed equation of evolution in time (unless the dynamics is harmonic).

Remark 44. Let us again come back to the unperturbed chain, and see the evolution equation for the Wigner transform in this case. The space and time scaled

$$\widehat{W}_\epsilon(t,p,k) := \widehat{W}(t/\epsilon, \epsilon p, k)$$

obeys

$$\frac{d}{dt}\widehat{W}_\epsilon(t,p,k) = \frac{i}{\epsilon}\left[\omega\left(t,k+\frac{\epsilon p}{2}\right) - \omega\left(t,k-\frac{\epsilon p}{2}\right)\right]\widehat{W}_\epsilon(t,p,k),$$

and the factor ϵ^{-1}, generating fast oscillation of the wave function, gets absorbed by the derivative of ω in the large scale.

2.3 Scaling limits: phonon Boltzmann equation

The evolution equations for the Wigner transform of perturbed system will be presented in chapters 3 and 4 for specific perturbations. In the regime of hyperbolic scaling

$t \to t/\epsilon$, $x \to x/\epsilon$, $\epsilon \ll 1$, the limit $\overline{u}$ of averaged Wigner transform of perturbed systems satisfies the linear phonon Boltzmann equation, which in one dimensional case has the form

$$\partial_t \bar{u}(t,x,k) + \frac{\omega'(k)}{2\pi}\partial_x \bar{u}(t,x,k) = \mathcal{L}\bar{u}(t,x,k). \tag{86}$$

Here $\omega(\cdot)$ is the dispersion relation and scattering operator $\mathcal{L}$ acting on variable k has the form

$$\mathcal{L}f(k) = \int_{\mathbb{T}} R(k,k')\left[f(k') - f(k)\right] dk'$$

with scattering kernel $R(k,k')$. Operator $\mathcal{L}$ generates a compound Poisson process on $\mathbb{T}$. Assume for a moment that there is no scattering operator in (86), so we have the following transport equation

$$\partial_t \bar{u}(t,x,k) + \frac{\omega'(k)}{2\pi}\partial_x \bar{u}(t,x,k) = 0. \tag{87}$$

Such equation emerges from deterministic dynamics of harmonic oscillators. Solution of (87) with initial condition $\bar{u}(0,x,k) = \bar{u}_0(x,k)$ is formally given by

$$\bar{u}(t,x,k) = \bar{u}_0\big(x - t\,\omega'(k)/(2\pi), k\big).$$

We deal with a transport at constant velocity $\omega'(k)/2\pi$ in spatial variable x, and wave number k is only a parameter here.

Now let us come back to the phonon Boltzmann equation (86). The scattering operator $\mathcal{L}$ appears as the result of the stochastic perturbation, and specific form of the kernel $R(k,k')$ depends on the type of it. $\mathcal{L}$ is the generator of compound Poisson process on the state space $\mathbb{T}$, such as described in section 1.9. If we denote by $K_t(k)$ the state, at moment t, of the process that started at k (i.e. such that $K_0(k) = k$), then solution of (86) is represented by the following formula ([28], p. 2278):

$$\bar{u}(t,x,k) = \mathbb{E}\bar{u}_0\left(x - \frac{1}{2\pi}\int_0^t \omega'(K_s(k))ds, K_t(k)\right).$$

A quasiparticle, being at $t = 0$ in initial state with wavenumber k, changes its states as a result of random collisions, finding itself in a state with wavenumber $K_t(k)$ at time $t > 0$. Hence the velocity at time t is given by $\omega'(K_t(k))/2\pi$ and

$$X_t(x) := x + \frac{1}{2\pi}\int_0^t \omega'(K_s(k))ds$$

is spatial position coordinate at t, given that the initial position was x. The Boltzmann equation (86) describes evolution of the density of Markov process (X_t, K_t) in the state space $\mathbb{R} \times \mathbb{T}$ ([13], p. 178, [9], p. 224).

2.4 Scaling limits: classical and fractional heat equation

Solution $u(t, x)$ of the classical heat equation

$$\partial_t u(t, x) = c\,\Delta u(t, x). \tag{88}$$

is interpreted as a temperature at time t at point with spatial coordinate x ([57], p. 5). Fourier's law satisfied by solution u states that ([19], p. 13)

$$q = -k\,\nabla u$$

where q is the rate of flow of heat energy through unit area per unit time, or *heat flux*, and positive proportionality coefficient k is called *conductivity*. Operator ∇ is the gradient in spatial variable x. In one spatial dimension the heat equation becomes

$$\partial_t u(t, x) = c\,\partial_x^2 u(t, x). \tag{89}$$

Positive constant c is called *thermal diffusivity* ([19], p. 15). Solution of the equation with appropriate initial condition $u(0, x) = u_0(x)$, assuming no boundary conditions, is given by the convolution

$$u(t, x) = \int_{\mathbb{R}} u_0(y)\,p_t(x - y)dy \tag{90}$$

with $p_t(x) = (4\pi ct)^{-1/2} \exp\{-x^2/(4ct)\}$. In terms of its probabilistic interpretation

$$u(t,x) = \mathbb{E} u_0(x + w(t)) \tag{91}$$

where $w(\cdot)$ is the Brownian motion with variance $\mathbb{E}(w(t))^2 = 2ct$. The Fourier transform $\widehat{u}(t,p)$ of $u(t,x)$ satisfies equation

$$\partial_t \widehat{u}(t,p) = -4c\pi^2 p^2 \widehat{u}(t,p).$$

It is given by the formula

$$\widehat{u}(t,p) = \widehat{u}_0(p)\, \mathrm{e}^{-t\psi(2\pi p)}, \tag{92}$$

here $\psi(p) = cp^2$ is the Lévy exponent of Brownian motion. Factor 2π in the argument $2\pi p$ appears because of the definition of Fourier transform we use here. If we put in (92) Lévy exponent $\tilde{\psi}(p) = c|p|^\alpha$ of symmetric α–stable distribution, $0 < \alpha < 2$, then we get the Fourier transform of solution of fractional diffusion equation

$$\partial_t u(t,x) = -c\big(-\partial_x^2\big)^{\alpha/2} u(t,x). \tag{93}$$

Here $\big(-\partial_x^2\big)^{\alpha/2}$ is fractional Laplacian: by definition, $\big(-\partial_x^2\big)^{\alpha/2} u(x)$ is the inverse Fourier transform of $(2\pi|p|)^\alpha\, \widehat{u}(p)$. Solution is given by (91) with $w(\cdot)$ replaced by symmetric α–stable process.

As shown in [28], under appropriate scaling, solution of the Boltzmann equation (86) converges to $u(t,x)$ which satisfies the fractional heat equation (93) . Solution $u(t,\cdot)$ is interpreted as the heat distribution along the body at time t. It is shown in [38] that solutions of (93) and (89), under proper scalings, are limits of microscopic model (80).

Chapter 3

Model with Brownian noise

The sources of results reported in this chapter are [13], [28] and [38].

3.1 The perturbation

Let us describe one dimensional system of harmonic oscillators with a weak stochastic noise introduced in [10, 11]. The dynamics are given by

$$\begin{cases} d\mathfrak{q}_y(t) = \mathfrak{p}_y(t)dt \\ d\mathfrak{p}_y(t) = -(\alpha * \mathfrak{q}(t))_y dt + d\zeta_y^{(\epsilon)}[\mathfrak{p}](t), \end{cases} \tag{94}$$

where

$$d\zeta_y^{(\epsilon)}[\mathfrak{p}](t) := \sqrt{\epsilon} \sum_{z=-1,0,1} (Y_{y+z}\mathfrak{p}_y) \circ dw_{y+z}(t), \tag{95}$$

$\epsilon \ll 1$. Here $\{w_y(t) : t \geq 0\}$, $y \in \mathbb{Z}$, are independent standard Brownian motions on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Stochastic differential on the right hand side is understood in Stratonovich sense, and operators Y_y, $y \in \mathbb{Z}$ are given by

$$Y_y := (\mathfrak{p}_y - \mathfrak{p}_{y+1})\partial_{\mathfrak{p}_{y-1}} + (\mathfrak{p}_{y+1} - \mathfrak{p}_{y-1})\partial_{\mathfrak{p}_y} + (\mathfrak{p}_{y-1} - \mathfrak{p}_y)\partial_{\mathfrak{p}_{y+1}}.$$

Component $\zeta_y^{(\epsilon)}$ is responsible for a random exchange of momenta between three adjacent oscillators located at $y-1$, y, $y+1$. Note that

$$Y_y(\mathfrak{p}_{y-1}^2 + \mathfrak{p}_y^2 + \mathfrak{p}_{y+1}^2) = Y_y(\mathfrak{p}_{y-1} + \mathfrak{p}_y + \mathfrak{p}_{y+1}) = 0,$$

so the vector field $\{Y_y\}$ is tangent to any circle given by equations

$$\begin{cases} \mathfrak{p}_{y-1}^2 + \mathfrak{p}_y^2 + \mathfrak{p}_{y+1}^2 = c_1, \\ \mathfrak{p}_{y-1} + \mathfrak{p}_y + \mathfrak{p}_{y+1} = c_2, \end{cases} \tag{96}$$

of constant energy c_1 and momentum c_2 of three adjacent oscillators. In consequence the total energy is preserved by the perturbation. The sum on the right hand side of (95) expands as follows

$$\begin{aligned} \sum_{z=-1,0,1} (Y_{y+z}\mathfrak{p}_y) \circ dw_{y+z}(t) = (\mathfrak{p}_{y-2} - \mathfrak{p}_{y-1}) \circ dw_{y-1}(t) \\ + (\mathfrak{p}_{y+1} - \mathfrak{p}_{y-1}) \circ dw_y(t) \\ + (\mathfrak{p}_{y+1} - \mathfrak{p}_{y+2}) \circ dw_{y+1}(t). \end{aligned}$$

The dynamics of $(\mathfrak{p}, \mathfrak{q})$ have the following Itô form

$$\begin{cases} d\mathfrak{q}_y(t) = \mathfrak{p}_y(t)dt \\ d\mathfrak{p}_y(t) = \left[-(\alpha * \mathfrak{q}(t))_y - \frac{\epsilon}{2}(\beta * \mathfrak{p}(t))_y\right] dt \\ \qquad + \sqrt{\epsilon} \sum_{j=-1,0,1} (Y_{y+j}\mathfrak{p}_y(t))dw_{y+j}(t), \quad y \in \mathbb{Z}. \end{cases} \tag{97}$$

where $\beta = \{\beta_y : y \in \mathbb{Z}\}$ is given by

$$\beta_y := \begin{cases} 6, & y = 0, \\ -2, & |y| = 1, \\ -1, & |y| = 2, \\ 0, & |y| > 2. \end{cases}$$

It is shown in [13] that the Wigner transform of the system under space and time scaling $t \to t/\epsilon$, $x \to x/\epsilon$ converges as $\epsilon \to 0$ to solution of phonon Boltzmann equation (86). Further investigation of this model in [28] shows that the limit obtained in [13] under superdiffusive scaling $t \to t/\epsilon$, $x \to x/\epsilon^{3/2}$, $\epsilon \to 0$ satisfies the fractional heat equation in the case that no pinning potential is present in the dynamics (94).

3.2 Convergence theorem

Asymptotics shown in [13] and [28] combined together indicate that Wigner transform of microscopic model (94) satisfies at some macroscopic scale fractional heat equation, if no pinning potential is present. On the other hand, if there is pinning potential, then classical heat equation is satisfied. The time–scaled wave function of the microscopic system is defined as

$$\widehat{\psi}^{(\epsilon)}(t,k) := \widehat{\psi}(\epsilon^{-1}t,k) = \omega(k)\widehat{\mathfrak{q}}(\epsilon^{-1}t,k) + i\widehat{\mathfrak{p}}(\epsilon^{-1}t,k).$$

Initial condition for the dynamics is random. Namely, we have a family of probability measures μ_ϵ, $\epsilon \in (0,1]$ on initial states of the system (at time $t = 0$). It is assumed in [13] that in the unpinned case

$$\lim_{R\to 0} \limsup_{\epsilon\to 0} \epsilon \int_{|k|<R} \langle |\widehat{\psi}_0(k)|^2\rangle_\epsilon \, dk = 0,$$

so that there is no concentration of energy at wave number $k = 0$ ([13], p. 177). Here the mean value with respect to the initial measure μ_ϵ is denoted by $\langle \cdot \rangle_\epsilon$. Analogous condition in [38], concerning scaling under which asymptotics of the microscopic dynamics resembles superdiffusive (or diffusive) heat transfer on the large scale, states that for some numbers $a, \gamma > 0$

$$K_{a,\gamma} := \limsup_{\epsilon\to 0} \epsilon^{1+\gamma} \int_{\mathbb{T}} \langle |\widehat{\psi}_0(k)|^2\rangle_\epsilon \frac{dk}{|k|^{2a}} < +\infty, \tag{98}$$

so the number a controls concentration of energy at $k = 0$. Number γ makes additional scaling factor. In [13], scaled Wigner transforms W_ϵ are defined by averaging, with respect to introduced probability measures, the following (random) Wigner transform

$$\frac{\epsilon}{2} \int_{\mathbb{R}} \mathrm{e}^{2\pi i p x}\, \widehat{\psi}\left(t, k + \frac{\epsilon p}{2}\right) \widehat{\psi}^*\left(t, k - \frac{\epsilon p}{2}\right) dp.$$

In [38], scaled and averaged Wigner and anti–Wigner transforms are defined by

$$W_{\epsilon,\gamma}(t,x,k) = \frac{\epsilon^{1+\gamma}}{2}\mathbb{E}_\epsilon \int_{\mathbb{R}} \mathrm{e}^{2\pi i p x}\,\widehat{\psi}\left(t, k+\frac{\epsilon^{1+\gamma}p}{2}\right)\widehat{\psi}^*\left(t, k-\frac{\epsilon^{1+\gamma}p}{2}\right)dp,$$

$$Y_{\epsilon,\gamma}(t,x,k) = \frac{\epsilon^{1+\gamma}}{2}\mathbb{E}_\epsilon \int_{\mathbb{R}} \mathrm{e}^{2\pi i p x}\,\widehat{\psi}\left(t, -k+\frac{\epsilon^{1+\gamma}p}{2}\right)\widehat{\psi}\left(t, k+\frac{\epsilon^{1+\gamma}p}{2}\right)dp,$$

wherein the mean value $\mathbb{E}_\epsilon$ is taken with respect to the product measure $\mu_\epsilon \otimes \mathbb{P}$.

Assuming either lack or presence of a pinning potential, limit of $W_{\epsilon,\gamma}(t)$ is solution $W(t,x)$ of either fractional heat equation

$$\partial_t W(t,x) = -\frac{\widehat{c}}{(2\pi)^{3/2}}(-\partial_x^2)^{3/4}W(t,x), \tag{99}$$

or classical heat equation

$$\partial_t W(t,x) = \frac{\widehat{c}}{(2\pi)^2}\partial_x^2 W(t,x). \tag{100}$$

Positive coefficients denoted by $\widehat{c}$ in the limiting equations are specific for each of the both cases and depend on the potential. It is assumed in [38] that the initial condition for either limiting equation (99) or (100) is $W(0,x) = W_0(x)$, where

$$W_0(x) = \int_{\mathbb{T}} W_0(x,k)dk,$$

and $W_0(x,k)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{T})$, or is the limit, in norm $\|\cdot\|_{a,b}$ defined by

$$\|J\|_{a,b} := \sup_p (1+p^2)^{b/2}\int_{\mathbb{T}} \frac{|J(p,k)|}{|k|^{2a}}dk, \tag{101}$$

of functions belonging to $\mathcal{S}(\mathbb{R}\times\mathbb{T})$. The potential $\alpha = \{\alpha(y) : y \in \mathbb{Z}\}$ in (97) is assumed to satisfy the following conditions

C1) $\alpha(y),\ y \in \mathbb{Z}$ are real numbers such that for some positive constants C, d we have $|\alpha(y)| \le C\mathrm{e}^{-d|y|}$, $y \in \mathbb{Z}$, C2) $\widehat{\alpha}(k)$ is real, $\widehat{\alpha}(k) > 0$ for $k \neq 0$ and if $\widehat{\alpha}(0) = 0$ then $\widehat{\alpha}''(0) > 0$,

see [13] on p. 2, and [38] on p. 6. Let us state the results. Theorem 2.1 in [38] says the following.

Theorem 45 ([38], p. 5)**.** *Assume that potential α satisfies conditions C1) and C2) stated above.*

(i) Let $\widehat{\alpha}(0) = 0$ (so there is no pinning potential), and let (98) *be satisfied for some $a \in (0, 1]$ and $\gamma \in (0, 2a/3)$. Furthermore, let the initial Wigner transform $W_{\epsilon,\gamma}(0)$ converges, as $\epsilon \to 0$, to some distribution $\widetilde{W}_0$ with finite norm* (101) *in the following sense: for every $\widetilde{J} \in \mathcal{S}$*

$$\lim_{\epsilon \to 0+} \langle W_{\epsilon,\gamma}(0), \widetilde{J} \rangle = \int_{\mathbb{R}\times\mathbb{T}} \widetilde{W}_0(x, k) \widetilde{J}^*(x, k) dx dk. \tag{102}$$

Then for every $\widetilde{J} \in \mathcal{S}$ and $t > 0$

$$\lim_{\epsilon \to 0+} \left\langle W_{\epsilon,\gamma}\left(\frac{t}{\epsilon^{3\gamma/2}}\right), \widetilde{J} \right\rangle = \int_{\mathbb{R}\times\mathbb{T}} W(t, x) \widetilde{J}^*(x, k) dx dk$$

where $W(t, x)$ satisfies fractional heat equation (99) *with the initial condition*

$$W(0, x) := \int_{\mathbb{T}} \widetilde{W}_0(x, k) dk, \tag{103}$$

and specific coefficient $\widehat{c} > 0$ depending on $\widehat{\alpha}''(0)$.

(ii) Assume that $\widehat{\alpha}(0) > 0$ (i.e. the pinning potential is present) and, with some $\gamma \in (0, 1/2)$ and $a \in (0, 1]$, (98) *is satisfied. Let $W_{\epsilon,\gamma}(0)$ be convergent to some distribution $\widetilde{W}_0$ with finite norm* (101) *in the sense of* (102)*. Then for every $\widetilde{J} \in \mathcal{S}$ and $t > 0$*

$$\lim_{\epsilon \to 0+} \left\langle W_{\epsilon,\gamma}\left(\frac{t}{\epsilon^{2\gamma}}\right), \widetilde{J} \right\rangle = \int_{\mathbb{R}\times\mathbb{T}} W(t, x) \widetilde{J}^*(x, k) dx dk,$$

where $W(t, x)$ is solution of heat equation (100) *with initial condition* (103) *and specific coefficient $\widehat{c} > 0$ depending on the potential.*

3.3 Equation for the Wigner transform

Let us take a look at the evolution equations of the wave function and the Wigner transform ([38], section 2.5). Denote $\mathrm{e}_y(k) := \mathrm{e}^{-2\pi iky}$. The stochastic differential in the Itô equation for $\widehat{\psi}^{(\epsilon)}$ is derived from the dynamics on lattice. Based on (97), equations for the dynamics of Fourier transforms $\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}$ are

$$\begin{cases} d\widehat{\mathfrak{q}}(t,k) = \widehat{\mathfrak{p}}(t,k)dt \\ d\widehat{\mathfrak{p}}(t,k) = -\left(\widehat{\alpha}(k)\widehat{\mathfrak{q}}(t,k) + \frac{\epsilon}{2}\widehat{\beta}(k)\widehat{\mathfrak{p}}(t,k)\right)dt \\ \qquad\qquad + \sqrt{\epsilon}\int_{\mathbb{T}} r(k,k')\,\widehat{\mathfrak{p}}(t,k-k')\,d\widehat{w}(t,k'). \end{cases} \tag{104}$$

Functions $\widehat{\beta}(k)$ and $r(k,k')$ in the formulas above are given by

$$\widehat{\beta}(k) := 8\sin^2(\pi k)\left[1 + 2\cos^2(\pi k)\right], \quad k \in \mathbb{T},$$

and

$$\begin{aligned} r(k,k') :=& \sin(2\pi k) + \sin[2\pi(k-k')] + \sin[2\pi(k'-2k)] \\ =& 4\sin(\pi k)\sin[\pi(k-k')]\sin\left[(2k-k')\pi\right], \quad k,k' \in \mathbb{T}. \end{aligned} \tag{105}$$

Stochastic measure $\widehat{w}(\cdot)$ is formally given by

$$d\widehat{w}(t,k') = \sum_{z\in\mathbb{Z}} \mathrm{e}^{-2\pi ik'y} w_y(t)dk'.$$

The wave function $\widehat{\psi}$ satisfies

$$\begin{aligned} d\widehat{\psi}(t,k) = \omega(k)\widehat{\mathfrak{p}}(t,k)dt - i\,\widehat{\alpha}(k)\widehat{\mathfrak{q}}(t,k) - \frac{i\epsilon}{2}\widehat{\beta}(k)\widehat{\mathfrak{p}}(t,k)dt \\ + \sqrt{\epsilon}\int_{\mathbb{T}} r(k,k')\,\widehat{\mathfrak{p}}(t,k-k')d\widehat{w}(t,k'). \end{aligned}$$

Since $\omega(\cdot)$ is even and $\mathfrak{p}$, $\mathfrak{q}$ are real, we have

$$\widehat{\psi}^*(-k) = \omega(k)\widehat{\mathfrak{q}}(k) - i\widehat{\mathfrak{p}}(k)$$

and

$$\widehat{\mathfrak{p}}(k) = -\frac{i}{2}\left(\widehat{\psi}(k) - \widehat{\psi}^*(-k)\right).$$

Hence, with initial condition $\widehat{\psi}_0 \in L_2(\mathbb{T})$, the evolution equation for $\widehat{\psi}$ reads

$$d\widehat{\psi}^{(\epsilon)}(t) = A_\epsilon[\widehat{\psi}^{(\epsilon)}(t)]dt + \sum_{y\in\mathbb{Z}} Q[\widehat{\psi}^{(\epsilon)}(t)](e_y)dw_y(t), \tag{106}$$
$$\widehat{\psi}^{(\epsilon)}(0) = \widehat{\psi}_0,$$

where mapping $A_\epsilon : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is defined by the formula

$$A_\epsilon[\phi](k) := -\frac{i}{\epsilon}\omega(k)\phi(k) - \frac{\widehat{\beta}(k)}{4}\left(\phi(k) - \phi^*(-k)\right), \tag{107}$$

and, for any $\phi \in L_2(\mathbb{T})$, $Q[\phi] : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is operator defined by

$$Q[\phi](g)(k) := i\int_{\mathbb{T}} r(k,k')\left(\phi(k-k') - \phi^*(k'-k)\right) g(k')dk'.$$

There exists unique solution of (106) with values in $L_2(\mathbb{T};\mathbb{C})$ (compare with equation (64)), and the conservation law holds fo the solution:

$$\|\widehat{\psi}^{(\epsilon)}(t)\|_{L_2(\mathbb{T})} = const. \tag{108}$$

for $t \geq 0$, $\mathbb{P}$–almost surely.

Regarding Wigner and anti–Wigner transform, let

$$\widehat{W}_\epsilon(t,p,k) := \left\langle \left(\widehat{\psi}^{(\epsilon)}\right)^* \left(t, k - \frac{\epsilon p}{2}\right) \widehat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2}\right) \right\rangle_\epsilon$$

and

$$\widehat{Y}_\epsilon(t,p,k) := \left\langle \widehat{\psi}^{(\epsilon)} \left(t, -k + \frac{\epsilon p}{2}\right) \widehat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2}\right) \right\rangle_\epsilon.$$

For $J \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ let

$$\|J\|_{\mathcal{A}} := \sup_{p\in\mathbb{R}} \int_{\mathbb{T}} |J(p,k)|\, dk.$$

It follows by the Hölder's inequality that

$$\int_{\mathbb{T}} |\widehat{W}_\epsilon(t,p,k)|\, dk \le \int_{\mathbb{T}} \left|\widehat{\psi}^{(\epsilon)}\left(t,k\right)\right|^2 dk,$$

and conservation law (108) together with assumption (98) implies that

$$\mathbb{E}\|\widehat{W}_\epsilon(t)\|_{\mathcal{A}} < \infty. \quad \text{Similarly,} \quad \mathbb{E}\|\widehat{Y}_\epsilon(t)\|_{\mathcal{A}} < \infty.$$

Also more general norms are used in [38]:

$$\|J\|_{\mathcal{A}_a} := \sup_p (1+p^2)^{-a/2} \int_{\mathbb{T}} |J(p,k)| dk,$$

$$\|J\|_{\mathcal{A}'_a} := \int_{\mathbb{R}} (1+p^2)^{a/2} \sup_k |J(p,k)| dp.$$

If we denote by $\mathcal{A}_a$ and $\mathcal{A}'_a$ a Banach spaces defined as closures of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in norms $\|\cdot\|_{\mathcal{A}_a}$, $\|\cdot\|_{\mathcal{A}'_a}$ respectively, then elements of $\mathcal{A}'_a$ can be considered as elements of dual of $\mathcal{A}_a$.

Stochastic evolution equation for $\widehat{W}_\epsilon(t)$ reads

$$\begin{aligned} d\widehat{W}_\epsilon(t,p,k) = &\Bigg\{ \left\langle (A_\epsilon[\widehat{\psi}^{(\epsilon)}])^* \left(t, k-\frac{\epsilon p}{2}\right) \widehat{\psi}^{(\epsilon)}\left(t, k+\frac{\epsilon p}{2}\right)\right\rangle_\epsilon \\ &+ \left\langle (\widehat{\psi}^{(\epsilon)})^* \left(t, k-\frac{\epsilon p}{2}\right) A_\epsilon[\widehat{\psi}^{(\epsilon)}]\left(t, k+\frac{\epsilon p}{2}\right)\right\rangle_\epsilon \\ &+ \sum_{j\in\mathbb{Z}} \left\langle (Q[\widehat{\psi}^{(\epsilon)}](e_j))^* \left(t, k-\frac{\epsilon p}{2}\right) Q[\widehat{\psi}^{(\epsilon)}](e_j)\left(t, k+\frac{\epsilon p}{2}\right)\right\rangle_\epsilon \Bigg\} dt \\ &+ d\mathcal{M}_t^{(\epsilon)}(p,k), \end{aligned} \tag{109}$$

here $\{\mathcal{M}_t^{(\epsilon)},\, t\ge 0\}$ with $\{\mathcal{F}_t,\, t\ge 0\}$ is a *local martingale* (see [20], p. 96)

$$\begin{aligned} \mathcal{M}_t^{(\epsilon)}(p,k) := &\sum_{j\in\mathbb{Z}} \int_0^t \left\langle (Q[\widehat{\psi}^{(\epsilon)}(s)](e_j)^* \left(k-\frac{\epsilon p}{2}\right) \widehat{\psi}^{(\epsilon)}\left(s, k+\frac{\epsilon p}{2}\right)\right\rangle_\epsilon dw_j(s) \\ &+ \sum_{j\in\mathbb{Z}} \int_0^t \left\langle (\widehat{\psi}^{(\epsilon)})^* \left(s, k-\frac{\epsilon p}{2}\right) Q[\widehat{\psi}^{(\epsilon)}(s)](e_j)\left(k+\frac{\epsilon p}{2}\right)\right\rangle_\epsilon dw_j(s). \end{aligned}$$

Denote $\overline{W}_\epsilon(t) := \mathbb{E}\widehat{W}_\epsilon(t)$ and $\overline{Y}_\epsilon(t) := \mathbb{E}\widehat{Y}_\epsilon(t)$. Application of the mean value with respect to the Brownian motions on the left and right side in (109) reveals that $\overline{W}_\epsilon$ satisfies the following equation

$$\langle \partial_t \overline{W}_\epsilon(t), J\rangle = \langle \overline{W}_\epsilon(t), (iB + \mathcal{L})\, J\rangle + \langle \mathcal{R}_\epsilon(t), J\rangle, \quad \forall J \in \mathcal{S}, \tag{110}$$

where B, $\mathcal{L}$ and $\mathcal{R}_\epsilon(t)$ are defined as follows. Operator B is given by

$$Bf(p,k) := p\,\omega'(k) f(p,k), \qquad f \in \mathcal{S}.$$

Bounded linear operator $\mathcal{L}$ acts on $f \in \mathcal{S}$ according to the formula

$$\mathcal{L}f(p,k) := 2\int_{\mathbb{T}} R(k,k')[f(p,k') - f(p,k)]dk', \quad f \in \mathcal{S}.$$

Here, with $r(\cdot,\cdot)$ given by (105),

$$\begin{aligned} R(k,k') :=& \frac{1}{2}\left[r^2(k,k-k') + r^2(k,k+k')\right] \\ =& 8\sin^2(\pi k)\sin^2(\pi k')\left\{\sin^2\left[\pi(k+k')\right] + \sin^2\left[\pi(k-k')\right]\right\}. \end{aligned}$$

Kernel $R(k,k')$ emerges as the limit of

$$R_\varepsilon(p,k,k') := \frac{1}{2}\left[\rho(k,k+k',\epsilon p) + \rho(k,k-k',\epsilon p)\right], \tag{111}$$

where

$$\rho(k,k',p) := r\left(k - \frac{p}{2}, k'\right) r\left(k + \frac{p}{2}, k'\right).$$

$\mathcal{R}_\epsilon(t)$ in (110) collects components that are negligible as $\epsilon \to 0$ (see [38] p. 9).

3.4 About the proof

In what follows we present some ideas of the proof given in [38]. We denote by $\widetilde{J}(x,k)$ the inverse Fourier transform in the variable p of function $J(p,k) \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$. Let $\overline{W}(t)$ satisfy Boltzmann equation (86). Further, let $\overline{U}(t)$ be the Fourier transform in variable x of $\overline{W}(t)$, so that the following relation holds

$$\langle \overline{U}(t), J\rangle = \langle \overline{W}(t), \widetilde{J}\rangle$$

for all $t \geq 0$ and $J \in \mathcal{S}$. The equation for $\overline{U}(t)$ obtained by applying the Fourier transform in x on the both sides of (86) is

$$\partial_t \overline{U}(t) - iB\overline{U}(t) = \mathcal{L}\overline{U}(t). \tag{112}$$

The solution of (112) with initial condition $\overline{U}(0) = \overline{U}_0 \in \mathcal{S}$ is defined as a continuous function $t \to \overline{U}(t)$ with values in function space $\mathcal{A}$, for every $J \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ satisfying

$$\langle \overline{U}(t), J \rangle - \langle \overline{U}_0, J \rangle = \int_0^t \langle \overline{U}(s), (iB + \mathcal{L})\, J \rangle \, ds. \tag{113}$$

The probabilistic representation of the solution is used. If we denote by $K_t(k)$ the state, at moment t, of the process that started at k, then the solution is represented by the following Feynman–Kac formula

$$\overline{U}(t,p,k) = \mathbb{E}\left[\exp\left\{-ip\int_0^t \omega'(K_s(k))ds\right\} \overline{U}_0(p, K_t(k))\right]. \tag{114}$$

Since the process K_t is reversible, the following holds

$$\int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\exp\left\{-ip\int_0^t \omega'(K_s(k))ds\right\} \overline{U}_0(p, K_t(k)) J^*(p, K_0(k))\right] dpdk$$

$$= \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\exp\left\{-ip\int_0^t \omega'(K_{t-s}(k))ds\right\} \overline{U}_0(p, K_0(k)) J^*(p, K_t(k))\right] dpdk$$

$$= \int_{\mathbb{R}\times\mathbb{T}} \overline{U}_0(p,k)\, \mathbb{E}\left[\exp\left\{ip\int_0^t \omega'(K_s(k))ds\right\} J(p, K_t(k))\right]^* dpdk.$$

Hence we have

$$\left\langle e^{(iB+\mathcal{L})t}\, \overline{U}_0, J \right\rangle = \left\langle \overline{U}(t), J \right\rangle = \left\langle \overline{U}_0, J(t) \right\rangle, \tag{115}$$

where

$$J(t,p,k) := \mathbb{E} \exp\left\{ip\int_0^t \omega'(K_s(k))ds\right\} J(p, K_t(k)). \tag{116}$$

Further, comparing (110) and (112) we see that the following *Duhamel formula* holds for $\overline{W}_\epsilon(t)$

$$\langle \overline{W}_\epsilon(t), J\rangle = \langle \overline{W}_\epsilon(0), J(t)\rangle + \int_0^t \langle \mathcal{R}_\epsilon(s), J(t-s)\rangle ds. \tag{117}$$

Denote by $W_0(p,k)$ the Fourier transform in x of $\widetilde{W}_0(x,k)$ appearing in Theorem (45), and let $\overline{W}_0(p) := \int_{\mathbb{T}} W_0(p,k)dk$. Let also $\overline{J}(p) := \int_{\mathbb{T}} J(p,k)dk$. The following estimate is one of steps in the proof of part (i) of Theorem (45) (see [38] on p. 11)

$$\left|\int_{\mathbb{R}\times\mathbb{T}} \left[\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(\frac{t}{\epsilon^{3\gamma/2}}, p\epsilon^\gamma, k\right) - \overline{W}_0(p)e^{-\widehat{c}|p|^{3/2}t}\right] J^*(p,k)dpdk\right| \tag{118}$$

$$\leq \left|\int_{\mathbb{R}\times\mathbb{T}} \left[\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(0, p\epsilon^\gamma, k\right) - W_0\left(p,k\right)\right] e^{-\widehat{c}|p|^{3/2}t}\overline{J}^*\left(p\right)dpdk\right| \tag{119}$$

$$+ R_s(\epsilon, t, J, W_0, K_{a,\gamma}),$$

with specified component R_s. Such estimate is obtained for any fixed $a \in (0,1]$ and $\gamma \in (0, 2a/3)$. For any $t_0 > 0$, R_s depends only on $\epsilon > 0$, $t \geq t_0$, $K_{a,\gamma}$ given by (98), and some norms of test function J and of W_0. It vanishes as $\epsilon \to 0$. The integral in line (119) also vanishes by assumption. As a result, (118) converges to zero. Now note that

$$\left\langle W_{\epsilon,\gamma}\left(\frac{t}{\epsilon^{3\gamma/2}}\right), \widetilde{J}\right\rangle = \frac{\epsilon^{1+\gamma}}{2}\int_{\mathbb{R}\times\mathbb{T}} \overline{W}_\epsilon\left(\frac{t}{\epsilon^{3\gamma/2}}, p\epsilon^\gamma, k\right) J^*(p,k)dpdk,$$

and that Fourier transform of the solution of fractional heat equation (99) with initial condition

$$W(0,x) = \int_{\mathbb{R}} \mathrm{e}^{2\pi ipx}\,\overline{W}_0(p)dp \tag{120}$$

is given by the formula

$$W(t,p) := e^{-\widehat{c}|p|^{3/2}t}\,\overline{W}_0\left(p\right).$$

Regarding part (ii) of Theorem (45), similar estimate is obtained, and it gives analogous conclusions. We have ([38], p. 13)

$$\left|\int_{\mathbb{R}\times\mathbb{T}} \left[\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(\frac{t}{\epsilon^{2\gamma}}, p\epsilon^\gamma, k\right) - \overline{W}_0(p)e^{-\widehat{c}p^2 t}\right] J^*(p,k)dpdk\right| \tag{121}$$

$$\leq \left|\int_{\mathbb{R}\times\mathbb{T}} \left[\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(0, p\epsilon^\gamma, k\right) - W_0\left(p,k\right)\right] e^{-\widehat{c}p^2 t}\bar{J}^*\left(p\right)dpdk\right|$$

$$+ R_d(\epsilon, t, J, W_0, K_{a,\gamma}).$$

As in (118), R_d is established for $t \geq t_0 > 0$. Here $a \in (0,1]$ and $\gamma \in (0, 1/2)$ are fixed. Under the square bracket in (121) we see difference of two components. One is the Fourier transform in x of the Wigner transform $W_{\epsilon,\gamma}$, appropriately scaled in time, and the other is Fourier transform of the solution of heat equation (89) with initial condition (120).

The scattering phenomenon described by equation (86), which manifests itself at the intermediate scale, appears in the proofs of both estimates sketched above. In the proof of (118), the difference (in square brackets) between Wigner transform and fractional heat distribution on the left hand side is divided into sum of two components:

(i) One component involves $\overline{W}_\epsilon(t)$ and scaled solution of the Boltzmann equation. It reads

$$\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(\frac{t}{\epsilon^{3\gamma/2}}, p\epsilon^\gamma, k\right) - \overline{U}_\epsilon\left(\frac{t}{\epsilon^{3\gamma/2}}, p\epsilon^\gamma, k\right), \tag{122}$$

where

$$\overline{U}_\epsilon(t,p,k) = \mathbb{E}\left[\exp\left\{-ip\int_0^t \omega'(K_s(k))ds\right\} W_0(\epsilon^{-\gamma}p, K_t(k))\right] \tag{123}$$

is the Feynman–Kac formula for solution of the Boltzmann equation (112) with initial condition $\overline{U}_\epsilon(0,p,k) = W_0(\epsilon^{-\gamma}p, k)$.

(ii) The other component is

$$\overline{U}_\epsilon\left(\frac{t}{\epsilon^{3\gamma/2}}, p\epsilon^\gamma, k\right) - \overline{W}_0(p)e^{-\widehat{c}|p|^{3/2}t}. \tag{124}$$

Likewise, in proof of (121) the difference between Wigner transform and classical heat distribution is divided into sum of two components:

$$\frac{\epsilon^{1+\gamma}}{2}\overline{W}_\epsilon\left(\frac{t}{\epsilon^{2\gamma}}, p\epsilon^\gamma, k\right) - \overline{U}_\epsilon\left(\frac{t}{\epsilon^{2\gamma}}, p\epsilon^\gamma, k\right) \tag{125}$$

where $\overline{U}_\epsilon(t,p,k)$ is, as before, given by (123), and

$$\overline{U}_\epsilon\left(\frac{t}{\epsilon^{2\gamma}}, p\epsilon^\gamma, k\right) - \overline{W}_0(p)\mathrm{e}^{-\widehat{c}p^2 t}. \tag{126}$$

3.5 Additive functionals on Markov chains

Convergence of solution of Boltzmann equation (86) to fractional heat distribution $u(t,x)$ satisfying (93) is obtained in [28] as an application of formulated and proven there noncentral limit theorem for scaled additive functionals on a Markov chain (see Theorem 3.1 in there). The limit of functionals on Markov chain is a Lévy stable process related to $u(t,x)$. The Markov chain is the skeleton chain of time–continuous Markov jump process K_t seen in formula (114), i.e. the sequence of successive states visited by K_t.

Here is a description. Let us define sequence τ_n, $n = 0, 1, 2\dots$ of independent, identically distributed exponential random variables with $\mathbb{E}\tau_0 = 1$ and let

$$R(k) := \int_{\mathbb{T}} R(k,k')dk', \quad k \in \mathbb{T},$$

where $R(\cdot,\cdot)$ is scattering kernel in generator $\mathcal{L}$ of K_t given by the formula

$$\mathcal{L}f(\cdot) = \int_{\mathbb{T}} R(\cdot,k')\left[f(k') - f(\cdot)\right]dk'.$$

Further, let $\{\xi_n, n \geq 0\}$ be a Markov chain independent of $\tau_n, n \geq 0$ with transition probability $P(k, dk') = R^{-1}(k)R(k, k')dk'$. We have

$$R(k) = 2\sin^2(\pi k)\,[2 + \cos(2\pi k)]\,, \tag{127}$$

and in particular $\min_{k\in\mathbb{T}} R^{-1}(k) > 0$. Define a sequence of random moments

$$t_n := \sum_{k=0}^{n-1} \frac{\tau_k}{R(\xi_k)}, \qquad n \geq 1.$$

Given that K_0 and ξ_0 have the same distribution, variable t_n for each n can be seen as the moment of n–th jump of process K_t. We define random variable n_t as the number of jumps of K_t up to the time t:

$$n_t := \max\{n : \, t_n \leq t\}.$$

The Markov process K_t is given by

$$K_t = \xi_{n_t}, \qquad t \geq 0,$$

see [21] p. 162-163. Hence $\int_0^t \omega'(K_s)ds$ in the formula (114) can be expressed as

$$\int_0^t \omega'(K_s)ds = \sum_{k=0}^{[n_t]-1} \frac{\tau_k\,\omega'(\xi_k)}{R(\xi_k)} + \omega'(\xi_{n_t})[t - t_{n_t}]. \tag{128}$$

The sum on the right hand side has random number of components. Let $\pi(dk)$ be invariant measure for K_t. Sum of averaged number of components $\tau_k\, R^{-1}(\xi_k)\,\omega'(\xi_k)$ is defined as

$$\sum_{k=0}^{[\overline{T}t]} \tau_k R^{-1}(\xi_k)\omega'(\xi_k). \tag{129}$$

where $\overline{T} := \int_{\mathbb{T}} R^{-1}(k)\pi(dk)$. Denote by $\mathrm{R}_\epsilon(t)$ the difference between scaled (128) and (129), namely

$$\mathrm{R}_\epsilon(t) = \epsilon^\gamma \int_0^{t/\epsilon^{\beta\gamma}} \omega'(K_s)ds - \epsilon^\gamma \sum_{k=0}^{[\overline{T}t/\epsilon^{\beta\gamma}]} \frac{\tau_k\,\omega'(\xi_k)}{R(\xi_k)}, \tag{130}$$

where $\beta = 3/2$ or $\beta = 2$ depending on lack or presence of the pinning potential. The first step in approaching $\overline{W}_0(p)e^{-\widehat{c}|p|^\beta t}$ by

$$\overline{U}_{\epsilon,\gamma}(t,p,k) = \mathbb{E}\left[\exp\left\{-ip\epsilon^\gamma \int_0^{t/\epsilon^{\beta\gamma}} \omega'(K_s(k))ds\right\} W_0(p, K_{t/\epsilon^{\beta\gamma}}(k))\right] \tag{131}$$

is to show that $\mathrm{R}_\epsilon(t)$ vanishes as $\epsilon \to 0$. By doing this, see section 6 in [28], additive functional on Markov process K_t is replaced with additive functional (129) on the Markov chain $\{\xi_n : n \geq 0\}$.

Further convergence of additive functional on Markov chain is an instance of general case formulated and proven in [28]. A Markov chain $X = \{X_n,\, n \geq 0\}$ in a Polish metric space (E, d) is considered. Let π be invariant probability measure for X. For a number $\beta \in (1, 2]$ and a function $\Psi : E \to \mathbb{R}$ satisfying $\int \Psi d\pi = 0$, for any $N \in \mathbb{N}$ the following process of scaled partial sums is defined:

$$Z_t^{(N)} := \frac{1}{N^{1/\beta}} \sum_{n=0}^{[Nt]} \Psi(\xi_n), \quad t \geq 0. \tag{132}$$

Under a set of additional assumptions limit theorems hold for (132). In particular:

– if $\alpha \in (1, 2)$, and for some numbers c_*^+,$c_*^- \geq 0$ such that $c_*^+ + c_*^- > 0$ we have

$$\lim_{\lambda\to\infty} \lambda^\alpha \pi(\Psi \geq \lambda) = c_*^+ \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda^\alpha \pi(\Psi \leq -\lambda) = c_*^-, \tag{133}$$

then, under some additional assumptions (for details we refer to Theorem 2.7 in [28] and Theorem 5.5 in [38]), $Z_t^{(N)}$ given by formula (132) with $\beta = \alpha$ converges to α–stable process as $N \to \infty$. The process in the limit has Lévy exponent

$$\psi(p) = \int_{\mathbb{R}} (\mathrm{e}^{ip\lambda} - 1 - ip\lambda)\nu(d\lambda),$$

with Lévy measure

$$\nu(d\lambda) = \frac{\alpha\, c_+^*}{\lambda^{1+\alpha}} \mathbb{1}_{(0,\infty)}(\lambda)d\lambda + \frac{\alpha\, c_-^*}{(-\lambda)^{1+\alpha}} \mathbb{1}_{(-\infty,0)}(\lambda)d\lambda.$$

– if $\Psi \in L_2(E, \pi)$ and some additional conditions are satisfied, see Theorem 5.8 in [38], (132) with $\beta = 2$ is convergent to Brownian motion as $N \to \infty$.

These results are applied to Markov chain $X_n = (\xi_n, \tau_n)$ with values in $\mathbb{T} \times \mathbb{R}_+$ and

$$\Psi(k, \tau) = \tau R^{-1}(k)\omega'(k). \tag{134}$$

The Markov chain $\{\xi_n : n = 0, 1 \dots\}$ has invariant probability measure

$$\pi(dk) = \theta R(k)dk, \quad \text{where} \quad \theta := \Big(\int_{\mathbb{T}} R(k)dk\Big)^{-1}$$

so the chain $\{(\xi_n, \tau_n) : n = 0, 1, \dots\}$ has invariant measure $\widetilde{\pi}(dk, d\tau)$ on $\mathbb{T} \times \mathbb{R}_+$ given by

$$\widetilde{\pi}(dk, d\tau) = \mathrm{e}^{-\tau}\theta R(k)\, dk\, d\tau.$$

In the unpinned case the dispersion is acoustic with $\omega(k) \sim k$ and $\omega'(k) \sim 1$ for positive $k \ll 1$. On the other hand $R(k) \sim k^2$ as $k \ll 1$, see (127). It follows that $\pi(\omega' R^{-1} \geq \lambda) \sim \lambda^{-3/2}$ as $\lambda \to \infty$. Furthermore the tails of $\Psi(k, \tau)$ given by (134) under $\widetilde{\pi}$ are of the same order as the tails of $R^{-1}(k)\omega'(k)$ under π, so

$$\widetilde{\pi}\,(\Psi \geq \lambda) \sim \lambda^{-3/2} \quad \text{as} \quad \lambda \to \infty.$$

Similarly $\widetilde{\pi}\,(\Psi \leq -\lambda) \sim \lambda^{-3/2}$ as $\lambda \to \infty$, and hence the superdiffusion limit with $\alpha = 3/2$.

In the pinned case $\omega'(k) \sim k$, ([38] p. 33), and $\pi(|\omega' R^{-1}| \geq \lambda) \sim \lambda^{-3}$. It follows that $\omega' R^{-1}$ belongs to $L_2(\pi)$ on $\mathbb{T}$.

Chapter 4

Model with Ornstein–Uhlenbeck perturbation

In this chapter we write about results obtained in [39]. Stochastic perturbation of the chain considered there is Markovian with space and time correlations.

4.1 The model and its hyperbolic scaling limit

System (2) takes the form

$$\begin{cases} \dfrac{d\mathfrak{q}_y(t)}{dt} = \mathfrak{p}_y(t) \\ \dfrac{d\mathfrak{p}_y(t)}{dt} = -(\alpha * \mathfrak{q}(t))_y + \sqrt{\epsilon} \displaystyle\sum_{k=-1,0,1} \left(Y_{y+k}\mathfrak{p}_y(t)\right) \xi_{y+k}(t), \quad y \in \mathbb{Z}, \end{cases} \tag{135}$$

where $\xi_y(t)$, $y \in \mathbb{Z}$ are Gaussian with covariance

$$\mathbb{E}\xi_y(t)\xi_z(s) = \int_{\mathbb{T}} \mathrm{e}^{-2\pi ik(y-z)} \mathrm{e}^{-\gamma(k)|t-s|} \sigma(k) dk.$$

Continuous (together with their first and second order derivatives) functions $\gamma(\cdot)$, $\sigma(\cdot)$ are real, even, σ is nonnegative and γ is strictly positive. Equations (135) differ from (97) only in replacement of stochastic (Stratonovich) differentials of Brownian motions $w_y(t)$ with differentials of $\xi_y(t)dt$. Vector field $\{Y_y\}$ is the same

$$Y_y := (\mathfrak{p}_y - \mathfrak{p}_{y+1})\partial_{\mathfrak{p}_{y-1}} + (\mathfrak{p}_{y+1} - \mathfrak{p}_{y-1})\partial_{\mathfrak{p}_y} + (\mathfrak{p}_{y-1} - \mathfrak{p}_y)\partial_{\mathfrak{p}_{y+1}}$$

and total energy is preserved under the perturbation. It is assumed that initial vectors $\mathfrak{p}$ and $\mathfrak{q}$ belong to $l_2(\mathbb{Z})$ and are random with law μ_ϵ satisfying

$$K := \sup_{\epsilon\in(0,1]} \epsilon\langle\mathcal{H}(\mathfrak{p},\mathfrak{q})\rangle_{\mu_\epsilon} < +\infty, \tag{136}$$

here $\langle\,\cdot\,\rangle_{\mu_\epsilon}$ is the mean value with respect to μ_ϵ and $\mathcal{H}(\mathfrak{p},\mathfrak{q})$ is the Hamiltonian (79). The result obtained in [39] states that under hyperbolic scaling $t \to t/\epsilon$, $x \to x/\epsilon$ the energy transport in the chain satisfies linear phonon Boltzmann equation

$$\partial_t\bar{u}(t,x,k) + \frac{\omega'(k)}{2\pi}\partial_x\bar{u}(t,x,k) = \int_{\mathbb{T}} R(k,k')\left[\bar{u}(t,x,k') - \bar{u}(t,x,k)\right]dk'.$$

The scattering kernel $R(k,k')$ is more complex than in the case with Brownian noise. We recall that in that case it was

$$R(k,k') = \frac{1}{2}R_+(k,k') + \frac{1}{2}R_-(k,k')$$

with

$$R_\pm(k,k') := 16\sin^2(\pi k)\sin^2(\pi k')\sin^2(\pi(k\mp k')),\quad k,k'\in\mathbb{T}.$$

This time it depends on parameters $\gamma(\cdot)$ and $\sigma(\cdot)$ of the Ornstein–Uhlenbeck perturbation and it is also dependent – in contrast to dynamics with Brownian noise – on the dispersion relation $\omega(\cdot)$. It is given by the formula

$$R(k,k') := \frac{2\sigma(k+k')\gamma(k+k')R_+(k,k')}{\gamma^2(k+k') + \left[\omega(k')+\omega(k)\right]^2} + \frac{2\sigma(k-k')\gamma(k-k')R_-(k,k')}{\gamma^2(k-k') + \left[\omega(k')-\omega(k)\right]^2}.$$

Let us outline the derivation of the Wigner transform and its dynamics for this model. First we write a more direct form of stochastic part of (135). We have

$$Y_{y-1}\mathfrak{p}_y = \mathfrak{p}_{y-2} - \mathfrak{p}_{y-1},\quad Y_y\mathfrak{p}_y = \mathfrak{p}_{y+1} - \mathfrak{p}_{y-1},\quad Y_{y+1}\mathfrak{p}_y = \mathfrak{p}_{y+1} - \mathfrak{p}_{y+2}$$

hence

$$\sum_{k=-1,0,1} Y_{y+k}\mathfrak{p}_y(t)\,\xi_{y+k}(t) = \sum_{|z|\le 2} \mathfrak{p}_{y+z}(t)\,\zeta_{y,z}(t),$$

where $\zeta_{y,0}(t) = 0$ and

$$\zeta_{y,-2}(t) := \xi_{y-1}(t), \quad \zeta_{y,2}(t) := -\xi_{y+1}(t),$$

$$\zeta_{y,-1}(t) := -\xi_y(t) - \xi_{y-1}(t), \quad \zeta_{y,1}(t) := \xi_y(t) + \xi_{y+1}(t).$$

The wave function is defined as earlier. According to dynamics (135) it satisfies the following system of equations

$$\frac{d\psi_y^{(\epsilon)}(t)}{dt} = -\frac{i}{\epsilon}\left(\widetilde{\omega} * \psi^{(\epsilon)}(t)\right)_y \qquad (137)$$

$$+ \frac{1}{2\sqrt{\epsilon}} \sum_{|z|\le 2}\left(\psi_{y+z}^{(\epsilon)}(t) - (\psi^{(\epsilon)})^*_{y+z}(t)\right)\zeta_{y,z}^{(\epsilon)}(t)$$

where $\zeta_{y,z}^{(\epsilon)}(t) := \zeta_{y,z}(t/\epsilon)$. We apply the Fourier transform to both sides of the equation and we get

$$\frac{d\widehat{\psi}^{(\epsilon)}(t,k)}{dt} = -\frac{i}{\epsilon}\omega(k)\widehat{\psi}^{(\epsilon)}(t,k)$$

$$+ \frac{i}{\sqrt{\epsilon}} \int_{\mathbb{T}} r(k,k')\left(\widehat{\psi}^{(\epsilon)}(t,k-k') - \widehat{\psi}^{(\epsilon)}(t,k'-k)^*\right)\widehat{\xi}^{(\epsilon)}\left(t,dk'\right).$$

The kernel $r(k,k')$ is the same as in the case of Brownian noise: with $\mathrm{e}^{-2\pi ikz}$ denoted by $\mathrm{e}_z(k)$ we have

$$\sum_{y\in\mathbb{Z}} \mathrm{e}^{-2\pi iky} \sum_{l=-2,2} \mathfrak{p}_{y+l}\zeta_{y,l} = \mathrm{e}_1(k)\left[(\mathrm{e}_1\widehat{\mathfrak{p}}) * \widehat{\xi}\,\right](k) - \mathrm{e}_{-1}(k)\left[(\mathrm{e}_{-1}\widehat{\mathfrak{p}}) * \widehat{\xi}\,\right](k)$$

$$= 2i\int_{\mathbb{T}} \sin\left(2\pi(k'-2k)\right)\widehat{\mathfrak{p}}(k-k')\widehat{\xi}(k')dk',$$

$$\sum_{y\in\mathbb{Z}} \mathrm{e}^{-2\pi iky} \sum_{l=-1,1} \mathfrak{p}_{y+l}\zeta_{y,l} = \left[\left((\mathrm{e}_{-1}-\mathrm{e}_1)\widehat{\mathfrak{p}}\right) * \widehat{\xi}\right](k) + (\mathrm{e}_{-1}-\mathrm{e}_1)\left[\widehat{\mathfrak{p}} * \widehat{\xi}\right](k)$$
$$= 2i\int_{\mathbb{T}} \left[\sin\left(2\pi(k-k')\right) + \sin\left(2\pi k\right)\right] \widehat{\mathfrak{p}}(k-k')\widehat{\xi}(k')dk',$$

and we get

$$r(k,k') = \sin\left(2\pi(k'-2k)\right) + \sin\left(2\pi(k-k')\right) + \sin\left(2\pi k\right)$$
$$= 4\sin(\pi k)\sin\left(\pi(k-k')\right)\sin\left(\pi(2k-k')\right).$$

4.2 Evolution of the Wigner transform

Wigner transform and anti–Wigner transform for (135) are defined as

$$W_\epsilon(t,x,k) = \int_{\mathbb{R}} \mathrm{e}^{2\pi ipx}\,\widehat{W}_\epsilon(t,p,k)\,dp \quad \text{and} \quad Y_\epsilon(t,x,k) = \int_{\mathbb{R}} \mathrm{e}^{2\pi ipx}\,\widehat{Y}_\epsilon(t,p,k)\,dp,$$

where

$$\widehat{W}_\epsilon(t,p,k) := \frac{\epsilon}{2}\left\langle \left(\widehat{\psi}^{(\epsilon)}\right)^* \left(t, k-\frac{\epsilon p}{2}\right)\widehat{\psi}^{(\epsilon)}\left(t,k+\frac{\epsilon p}{2}\right)\right\rangle_{\mu_\epsilon},$$

$$\widehat{Y}_\epsilon(t,p,k) := \frac{\epsilon}{2}\left\langle \widehat{\psi}^{(\epsilon)}\left(t,-k+\frac{\epsilon p}{2}\right)\widehat{\psi}^{(\epsilon)}\left(t,k+\frac{\epsilon p}{2}\right)\right\rangle_{\mu_\epsilon}$$

for $(p,k)\in\mathbb{R}\times\mathbb{T}$ and $t\geq 0$. Energy conservation and the assumption (136) on initial condition imply that almost surely

$$\sup_{t,\epsilon}\|\widehat{W}_\epsilon(t)\|_{\mathcal{A}} \leq K \quad \text{and} \quad \sup_{t,\epsilon}\|\widehat{Y}_\epsilon(t)\|_{\mathcal{A}} \leq K, \tag{138}$$

we recall that norm $\|\cdot\|_{\mathcal{A}}$ is defined by

$$\|J\|_{\mathcal{A}} := \sup_{p\in\mathbb{R}}\int_{\mathbb{T}} |J(p,k)|dk.$$

$\widehat{W}_\epsilon$ and $\widehat{Y}_\epsilon$ can be represented as the series

$$\widehat{W}_\epsilon(t,p,k) = \sum_{y,y'\in\mathbb{Z}} \mathcal{W}^\epsilon_{y,y'}(t,k)\mathrm{e}^{-\pi i\epsilon p(y+y')}$$

and

$$\widehat{Y}_\epsilon(t,p,k) = \sum_{y,y'\in\mathbb{Z}} \mathcal{Y}^\epsilon_{y,y'}(t,k)\mathrm{e}^{-\pi i\epsilon p(y+y')},$$

where

$$\mathcal{W}^\epsilon_{y,y'}(t,k) := \frac{\epsilon}{2}\big\langle\psi^{(\epsilon)}_y(t)\psi^{(\epsilon)}_{y'}(t)^*\big\rangle_{\mu_\epsilon}\mathrm{e}^{2\pi ik(y'-y)},$$
$$\mathcal{Y}^\epsilon_{y,y'}(t,k) := \frac{\epsilon}{2}\big\langle\psi^{(\epsilon)}_y(t)\psi^{(\epsilon)}_{y'}(t)\big\rangle_{\mu_\epsilon}\mathrm{e}^{2\pi ik(y'-y)},$$

for $y,y'\in\mathbb{Z}$, $k\in\mathbb{T}$. Since the system of equations for the pair $\mathcal{W}^\epsilon(t)$, $\mathcal{Y}^\epsilon(t)$ is not closed, the following distribution is also introduced

$$\mathcal{Y}^{\epsilon,-}_{y,y'}(t,k) := \mathcal{Y}^\epsilon_{y,y'}(t,-k)^*.$$

Evolution equations for $\mathcal{W}^\epsilon$ an $\mathcal{Y}^\epsilon$ are derived as follows. By (137) and Leibniz formula for differentiation:

$$\begin{aligned}
\partial_t \left\langle \psi^{(\epsilon)}_y(t)\big(\psi^{(\epsilon)}_{y'}\big)^*(t)\right\rangle_{\mu_\epsilon} &= \\
&= \Big\langle \frac{i}{\epsilon}\psi^{(\epsilon)}_y(t)\big(\tilde\omega * \psi^{(\epsilon)}_{y'}\big)^*(t) - \frac{i}{\epsilon}\big(\psi^{(\epsilon)}_{y'}\big)^*(t)\,\tilde\omega * \psi^{(\epsilon)}_y(t) \\
&\quad + \frac{1}{2\sqrt{\epsilon}}\sum_{|z|\le 2}\zeta^{(\epsilon)}_{y,z}(t)\left(\psi^{(\epsilon)}_{y+z}(t) - \big(\psi^{(\epsilon)}\big)^*_{y+z}(t)\right)\big(\psi^{(\epsilon)}_{y'}\big)^*(t) \qquad (139) \\
&\quad + \frac{1}{2\sqrt{\epsilon}}\sum_{|z|\le 2}\zeta^{(\epsilon)}_{y',z}(t)\left(\big(\psi^{(\epsilon)}\big)^*_{y'+z}(t) - \psi^{(\epsilon)}_{y'+z}(t)\right)\psi^{(\epsilon)}_y(t)\Big\rangle_{\mu_\epsilon}.
\end{aligned}$$

For appropriate sequences $J = \{J_{y,y'}(\cdot) : y,y'\in\mathbb{Z}\}$ of functions $J_{y,y'}(k)$ on $\mathbb{T}$ we define pairing of $\mathcal{W}^\epsilon$ with J by

$$\langle\mathcal{W}^\epsilon, J\rangle = \sum_{y,y'\in\mathbb{Z}}\int_{\mathbb{T}}\mathcal{W}^\epsilon_{y,y'}(k)J^*_{y,y'}(k)\,dk. \qquad (140)$$

Let $G(\cdot,\cdot)$ be a function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{T})$, and let a test function $G_\epsilon = \{G_{\epsilon,y,y'}(\cdot) : y,y'\in\mathbb{Z}\}$ be of the form

$$G_{\epsilon,y,y'}(k) := G(\epsilon(y+y')/2,k), \quad y,y'\in\mathbb{R},\ k\in\mathbb{T}. \qquad (141)$$

Putting G_ϵ into (140) we get

$$\langle \mathcal{W}^\epsilon, G_\epsilon\rangle = \frac{\epsilon}{2}\Big\langle \sum_{y,y'\in\mathbb{Z}}\int_{\mathbb{T}} \psi_y^{(\epsilon)}(t)\psi_{y'}^{(\epsilon)}(t)^* \mathrm{e}^{2\pi ik(y'-y)}\, G^*(\epsilon(y+y')/2,k)\,dk\Big\rangle_{\mu_\epsilon}$$

and the series averaged with respect to μ_ϵ on the right hand side rewrites as

$$\sum_{y,y'\in\mathbb{Z}}\int_{\mathbb{T}} \psi_y^{(\epsilon)}(t)\mathrm{e}^{-2\pi iky}\left[\psi_{y'}^{(\epsilon)}(t)\mathrm{e}^{-2\pi iky'}\right]^* \int_{\mathbb{R}} \mathrm{e}^{-\pi i\epsilon p(y+y')}\widehat{G}^*(p,k)\,dpdk$$

$$= \int_{\mathbb{R}}\int_{\mathbb{T}} \psi^{(\epsilon)}\Big(t,k+\frac{\epsilon p}{2}\Big)\left[\psi^{(\epsilon)}\Big(t,k-\frac{\epsilon p}{2}\Big)\right]^* \widehat{G}^*(p,k)\,dkdp,$$

so

$$\langle W_\epsilon, G\rangle = \langle \mathcal{W}^\epsilon, G_\epsilon\rangle, \tag{142}$$

where

$$\langle W_\epsilon, G\rangle = \int_{\mathbb{R}}\int_{\mathbb{T}} W_\epsilon(x,k)\,G^*(x,k)\,dkdx.$$

Let us present evolution equation for $\langle W_\epsilon, J\rangle$. We will denote stochastic integral $\int_{\mathbb{T}} f(k)\widehat{\xi}(t,dk)$ by $\mathcal{I}_{\xi(t)}f$. We also denote

$$r_{y,y'}^{(+)}(k,k') := r\left(k,k'\right)\mathrm{e}^{2\pi ik'y'}, \quad r_{y,y'}^{(-)}(k,k') := r\left(k,k'\right)\mathrm{e}^{2\pi ik'y}. \tag{143}$$

Given that $\frac{d}{dt}\langle \mathcal{W}^\epsilon(t), J\rangle = \langle \partial_t\mathcal{W}^\epsilon(t), J\rangle$, the following equation is derived from (139)

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{W}^\epsilon(t), J\rangle = &-\frac{1}{\epsilon}\langle \mathcal{W}^\epsilon(t), \mathfrak{D}J\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)}J\big\rangle \\ &-\frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^\epsilon, i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)}J\big\rangle - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon,-}, i\mathcal{K}^{(-)}_{\xi^{(\epsilon)}(t)}J\big\rangle.\end{aligned} \tag{144}$$

Here for any $J\in\mathcal{S}$

$$\mathfrak{D}J_{y,y'}(k) := i\sum_{z,z'} J_{z,z'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi ip(z-y)}\mathrm{e}^{2\pi ip'(z'-y')}\delta\omega(p,p',k)dpdp',$$

where $\delta\omega(p,p',k) := [\omega(k+p') - \omega(k-p)]$,

$$\mathcal{K}_\xi^{(-)}J_{y,y'}(k) := \sum_{z,z'} J_{z,z'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi ip(z-y)}\mathrm{e}^{2\pi ip'(z'-y')}\mathcal{I}_\xi r_{y,y'}^{(-)}(-k+p)dpdp',$$

$$\mathcal{K}_\xi^{(+)}J_{y,y'}(k) := \sum_{z,z'} J_{z,z'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi ip(z-y)}\mathrm{e}^{2\pi ip'(z'-y')}\mathcal{I}_\xi r_{y,y'}^{(+)}(k+p')dpdp',$$

and

$$\mathcal{K}^{(o)} := \mathcal{K}^{(-)} + \mathcal{K}^{(+)}.$$

$\mathcal{K}^{(+)}$ and $\mathcal{K}^{(-)}$ are stochastic as they depend on realization of $\xi^{(\epsilon)}$ by

$$\mathcal{I}_\xi r^{(+)}_{y,y'}(k) := \int_{\mathbb{T}} r^{(+)}_{y,y'}\left(k,k'\right)\widehat{\xi}(dk'), \quad \mathcal{I}_\xi r^{(-)}_{y,y'}(k) := \int_{\mathbb{T}} r^{(-)}_{y,y'}\left(k,k'\right)\widehat{\xi}(dk'), \quad (145)$$

where $\xi = \{\xi_z : z \in \mathbb{Z}\}$ is random, having the stationary measure of $\xi^{(\epsilon)}$ as the law. Equation for $\mathcal{Y}^\epsilon$ is as follows

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{Y}^\epsilon(t), J\rangle = \frac{1}{\epsilon}\big\langle \mathcal{Y}^\epsilon, i\Theta J\big\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^\epsilon(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)} J\big\rangle \\ - \frac{1}{\sqrt{\epsilon}}\left\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)} J_{\mathrm{e}}\right\rangle.\end{aligned}$$

Here

$$\Theta J_{y,y'}(k) := \sum_{z,z'} J_{z,z'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi i p(z-y)}\mathrm{e}^{2\pi i p'(z'-y')}\theta\omega(p,p',k)dpdp',$$

with $\theta\omega(p,p',k) := [\omega(k+p') + \omega(k-p)]$, and

$$(J_{\mathrm{e}})_{y,y'}(k) := J_{y,y'}(k) + J_{y',y}(-k).$$

If we replace the test function J in the above equations for $\mathcal{W}^\epsilon$ and $\mathcal{Y}^\epsilon$ by G^ϵ of the form (141), and use relation (142), we obtain equations for W_ϵ and Y_ϵ which are as follows. Equation for $W_\epsilon(t)$ reads

$$\begin{aligned}\frac{d}{dt}\langle W_\epsilon(t), G\rangle = -\left\langle \mathcal{W}^\epsilon(t), \mathfrak{D}_\epsilon G_\epsilon\right\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)} G_\epsilon\big\rangle \\ - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^\epsilon, i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)} G_\epsilon\big\rangle - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon,-}, i\mathcal{K}^{(-)}_{\xi^{(\epsilon)}(t)} G_\epsilon\big\rangle,\end{aligned} \quad (146)$$

where $\mathfrak{D}_\epsilon G := \frac{1}{\epsilon}\mathfrak{D}G$, and we have

$$\mathfrak{D}_\epsilon G_{\epsilon,y,y'}(k) = \frac{i}{\epsilon}\sum_z G(\epsilon z/2,k)\int_{\mathbb{T}} \mathrm{e}^{2\pi i p(z-y-y')}\delta\omega(k,p)dp,$$

$$\mathcal{K}^{(-)}_\xi G_{\epsilon,y,y'}(k) = \sum_z G(\epsilon z/2,k)\int_{\mathbb{T}} \mathrm{e}^{2\pi i p(z-y-y')}\mathcal{I}_\xi r^{(-)}_{y,y'}(-k+p)dp,$$

$$\mathcal{K}^{(+)}_\xi G_{\epsilon,y,y'}(k) = \sum_z G(\epsilon z/2,k)\int_{\mathbb{T}} \mathrm{e}^{2\pi i p(z-y-y')}\mathcal{I}_\xi r^{(+)}_{y,y'}(k+p)dp,$$

where $\delta\omega(k,p) := [\omega(k+p) - \omega(k-p)]$. Evolution of $Y_\epsilon(t)$ is given by

$$\frac{d}{dt}\langle Y_\epsilon(t), G\rangle = \frac{1}{\epsilon}\big\langle \mathcal{Y}^\epsilon, i\Theta G_\epsilon\big\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^\epsilon(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)} G_\epsilon\big\rangle \\ - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)} G_{\epsilon,\mathrm{e}}\big\rangle,$$

where $(G_{\epsilon,\mathrm{e}})_{y,y'}(k) := G(\epsilon(y+y')/2, k) + G(\epsilon(y+y')/2, -k)$. We have

$$\Theta G_{\epsilon,y,y'}(k) = \sum_z G(\epsilon z/2, k)\int_{\mathbb{T}} \mathrm{e}^{2\pi i p(z-y-y')}\theta\omega(k,p)dp,$$

here $\theta\omega(k,p) := [\omega(k+p) + \omega(k-p)]$.

4.3 About the proof: perturbed test function method

Calculations we present here are purely formal. They are correct in appropriate regime and for details we refer to the source [39]. Our purpose is to sketch the idea of the proof and see how the scattering in the limit emerges from the randomness at the microscopic scale. In particular we omit the description of functional spaces in which all the formal operations we cite are true in a strict sense.

Without OU perturbation there would be no random operators $\mathcal{K}^{(+)}$, $\mathcal{K}^{(-)}$, $\mathcal{K}^{(o)}$ in (146). These operators give rise to a jump process on $\mathbb{T}$. On the other hand, the deterministic term $\langle \mathcal{W}_\epsilon(t), \mathfrak{D}_\epsilon G^\epsilon\rangle$, should produce the operator $\omega'(k)\partial_x$ in the Boltzmann equation. We expand

$$\langle \mathcal{W}^\epsilon(t), \mathfrak{D}_\epsilon G_\epsilon\rangle = -\frac{i}{2}\Big\langle \sum_{z,y,y'\in\mathbb{Z}} \int_{\mathbb{T}} \psi_y^{(\epsilon)}(t)\psi_{y'}^{(\epsilon)}(t)^* \mathrm{e}^{2\pi i k(y'-y)}\, G^*(\epsilon z/2, k)\times \\ \times \int_{\mathbb{T}} \mathrm{e}^{2\pi i p(y+y'-z)}[\omega(k+p) - \omega(k-p)]dpdk\Big\rangle_{\mu_\epsilon}$$

and change variables $p \to -\epsilon p/2$ so we get that the right hand side is equal to

$$i \int_{-1/(2\epsilon)}^{1/(2\epsilon)} \int_{\mathbb{T}} \frac{\epsilon}{2} \Big\langle \widehat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2}\right) \widehat{\psi}^{(\epsilon)} \left(t, k - \frac{\epsilon p}{2}\right) \Big\rangle_{\mu_\epsilon} \times$$
$$\frac{\epsilon}{2} \sum_{z \in \mathbb{Z}} G^*(\epsilon z/2, k) e^{\pi i \epsilon p z} \frac{1}{\epsilon} \left[\omega \left(k + \frac{\epsilon p}{2}\right) - \omega \left(k - \frac{\epsilon p}{2}\right)\right] dpdk.$$

As $\epsilon \to 0$, $\frac{\epsilon}{2} \sum_{z \in \mathbb{Z}} G^*(\epsilon z/2, k) \mathrm{e}^{\pi i \epsilon p z}$ approaches $\widehat{G}^*(p, k)$. It follows that

$$\langle \mathcal{W}^\epsilon(t), \mathfrak{D}_\epsilon G_\epsilon \rangle = i \int_{-1/(2\epsilon)}^{1/(2\epsilon)} \int_{\mathbb{T}} \delta_\epsilon \omega(k, p) \widehat{W}_\epsilon(t, p, k) \widehat{G}^*(p, k) dpdk + O(t, \epsilon),$$

where
$$\delta_\epsilon \omega(k, p) := \frac{1}{\epsilon} \left[\omega \left(k + \frac{\epsilon p}{2}\right) - \omega \left(k - \frac{\epsilon p}{2}\right)\right]$$
and $O(t, \epsilon)$ converges to zero uniformly in t on any finite interval $[0, T]$.

Calculations related to random operators are much more sophisticated. The proof in [39] uses *perturbed test function* method. A random test function is introduced, which depends on the perturbation $\xi(t)$. As the closed evolution equation is formulated for the triple $(\mathcal{W}^\epsilon, \mathcal{Y}^\epsilon, \mathcal{Y}^{\epsilon,-})$, also the triple of perturbed test functions is defined, $(\bar{G}_0^\epsilon, \bar{G}_+^\epsilon, \bar{G}_-^\epsilon)$, with the following structure

$$\bar{G}_0^\epsilon(\xi) = \bar{G}^\epsilon + \sqrt{\epsilon}\, \bar{G}_0^{1,\epsilon}(\xi) + \epsilon\, \bar{G}_0^{2,\epsilon}(\xi), \tag{147}$$

$$\bar{G}_+^\epsilon(\xi) = \sqrt{\epsilon}\, \bar{G}_+^{1,\epsilon}(\xi) + \epsilon\, \bar{G}_+^{2,\epsilon}(\xi) \quad \text{and} \quad \bar{G}_-^\epsilon(\xi) = \sqrt{\epsilon}\, \bar{G}_-^{1,\epsilon}(\xi) + \epsilon\, \bar{G}_-^{2,\epsilon}(\xi),$$

where $\bar{G}^\epsilon$ is deterministic and the remaining components depend on realization of $\xi(t)$. They are specified in a way that they cancel out some components in the equations.

Denote by $\mathfrak{Q}$ the infinitesimal generator of the OU process $\{\xi(t)\}$ on $L_2(\mu_\sigma)$. It acts on the stochastic integral $\mathcal{I}u = \int_{\mathbb{T}} u(k)\, \widehat{\xi}(dk)$, $u \in L_2(\sigma)$, as follows

$$\mathfrak{Q}\mathcal{I}u = -\mathcal{I}\gamma u = -\int_{\mathbb{T}} \gamma(k) u(k)\, \widehat{\xi}(dk).$$

The differentiation of the following process

$$\Big\{ \Big\langle \mathcal{W}^{\epsilon}(t), G\big(\xi^{(\epsilon)}(t)\big) \Big\rangle : t \geq 0 \Big\}, \tag{148}$$

for an adequate test function G depending on $\xi(t)$, obeys formal rules of product differentiation. *Pseudogenerator* of process (148) (with appropriate function G), is denoted by $\mathfrak{L}$. We abbreviate $G^{\epsilon}(t) := G^{\epsilon}(\xi(t))$ and using (144) we get formulas

$$\begin{aligned} \mathfrak{L}\langle \mathcal{W}^{\epsilon}(t), \bar{G}_0^{\epsilon}(t)\rangle = & -\frac{1}{\epsilon}\langle \mathcal{W}^{\epsilon}(t), \mathfrak{D}\bar{G}_0^{\epsilon}(t)\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^{\epsilon}(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{\epsilon}(t)\big\rangle \\ & -\frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon}(t), i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{\epsilon}(t)\big\rangle - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon,-}(t), i\mathcal{K}^{(-)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{\epsilon}(t)\big\rangle \\ & + \frac{1}{\epsilon}\big\langle \mathcal{W}^{\epsilon}(t), \mathfrak{Q}\bar{G}_0^{\epsilon}(t)\big\rangle. \end{aligned} \tag{149}$$

Similarly

$$\begin{aligned} \mathfrak{L}\langle \mathcal{Y}^{\epsilon}(t), \bar{G}_+^{\epsilon}(t)\rangle = & \frac{1}{\epsilon}\big\langle \mathcal{Y}^{\epsilon}(t), i\Theta\bar{G}_+^{\epsilon}(t)\big\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon}(t), i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}_+^{\epsilon}(t)\big\rangle \\ & - \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^{\epsilon}(t), i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}_{+,\mathrm{e}}^{\epsilon}(t)\big\rangle + \frac{1}{\epsilon}\big\langle \mathcal{Y}^{\epsilon}(t), \mathfrak{Q}\bar{G}_+^{\epsilon}(t)\big\rangle. \end{aligned}$$

We note that (145) are Gaussian random variables with mean zero on probability space $(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$. Hence, for deterministic J, $\mathcal{K}_{\xi}^{(\cdot)} J$ have mean value zero with respect to the measure μ_σ. For $J = J(\xi)$, being itself a random variable on $(H_\lambda, \mathcal{B}(H_\lambda), \mu_\sigma)$, generalized operators $\widetilde{\mathcal{K}}^{(+)}, \widetilde{\mathcal{K}}^{(-)}, \widetilde{\mathcal{K}}^{(o)}$ are defined by

$$\widetilde{\mathcal{K}}J := \mathcal{K}J - \overline{\mathcal{K}J}, \quad \text{where} \quad \overline{\mathcal{K}J} := \int_{H_\lambda} \mathcal{K}J \, d\mu_\sigma, \tag{150}$$

so $\int_{H_\lambda} \widetilde{\mathcal{K}}J \, d\mu_\sigma = 0$. By expanding test functions according to (147), the sum

$$\mathfrak{L}\langle \mathcal{W}^{\epsilon}(t), \bar{G}_0^{\epsilon}(t)\rangle + \mathfrak{L}\langle \mathcal{Y}_{\epsilon}(t), \bar{G}_+^{\epsilon}(t)\rangle + \mathfrak{L}\langle \mathcal{Y}_{\epsilon}^{-}(t), \bar{G}_-^{\epsilon}(t)\rangle$$

expands into a long formula. Using abbreviations we present it as

$$\big\langle \mathcal{W}^\epsilon(t), \mathfrak{G}^\epsilon \big\rangle + \big\langle \mathcal{Y}^\epsilon(t), \mathfrak{G}^\epsilon_+ \big\rangle + \big\langle \mathcal{Y}^{\epsilon,-}(t), \mathfrak{G}^\epsilon_- \big\rangle \tag{151}$$

$$+ \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{W}^\epsilon(t), \mathfrak{G}^\epsilon_1(t) \big\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^\epsilon(t), \mathfrak{G}^\epsilon_{1,+}(t) \big\rangle + \frac{1}{\sqrt{\epsilon}}\big\langle \mathcal{Y}^{\epsilon,-}(t), \mathfrak{G}^\epsilon_{1,-}(t) \big\rangle$$

$$+ \big\langle \mathcal{W}^\epsilon(t), \mathfrak{G}^\epsilon_2(t) \big\rangle + \big\langle \mathcal{Y}^\epsilon(t), \mathfrak{G}^\epsilon_{2,+}(t) \big\rangle + \big\langle \mathcal{Y}^{\epsilon,-}(t), \mathfrak{G}^\epsilon_{2,-}(t) \big\rangle$$

$$+ \sqrt{\epsilon}\big\langle \mathcal{W}^\epsilon(t), \mathfrak{G}^\epsilon_3(t) \big\rangle + \sqrt{\epsilon}\big\langle \mathcal{Y}^\epsilon(t), \mathfrak{G}^\epsilon_{3,+}(t) \big\rangle + \sqrt{\epsilon}\big\langle \mathcal{Y}^{\epsilon,-}(t), \mathfrak{G}^\epsilon_{3,-}(t) \big\rangle,$$

and we will gradually explain what we have denoted by $\mathfrak{G}^\epsilon$, $\mathfrak{G}^\epsilon_+$ etc. except of $\mathfrak{G}^\epsilon_3(t)$, $\mathfrak{G}^\epsilon_{3,-}(t)$, $\mathfrak{G}^\epsilon_{3,+}(t)$ in the last line which, as shown in [39], vanishes as $\epsilon \to 0$. In the second line of (151) we have

$$\mathfrak{G}^\epsilon_1(t) := -i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t) + (\mathfrak{Q} - \mathfrak{D})\bar{G}^{1,\epsilon}_0(t),$$

$$\mathfrak{G}^\epsilon_{1,+}(t) := i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t) + (\mathfrak{Q} + i\Theta)\bar{G}^{1,\epsilon}_+(t),$$

$$\mathfrak{G}^\epsilon_{1,-}(t) := i\mathcal{K}^{(-)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t) + (\mathfrak{Q} - i\Theta)\bar{G}^{1,\epsilon}_-(t).$$

We did not yet specified $G^{1,\epsilon}_0(t), G^{1,\epsilon}_+(t), G^{1,\epsilon}_-(t)$ and we do this now. They are defined as solutions of the following equations

$$(\mathfrak{Q} - \mathfrak{D})\bar{G}^{1,\epsilon}_0(t) = -i\mathcal{K}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t), \tag{152}$$

$$(\mathfrak{Q} + i\Theta)\bar{G}^{1,\epsilon}_+(t) = i\mathcal{K}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t),$$

$$(\mathfrak{Q} - i\Theta)\bar{G}^{1,\epsilon}_-(t) = i\mathcal{K}^{(-)}_{\xi^{(\epsilon)}(t)}\bar{G}^\epsilon(t),$$

so that the second line in (151) vanishes μ_σ–almost surely. Let us derive a formula for $\bar{G}^{1,\epsilon}_0$. For a function $f(p,p',k)$ on $\mathbb{T}^3$ we denote

$$\mathfrak{F}[f]J_{y,y'}(k) := \sum_{z,z'} J_{z,z'}(k)\int_{\mathbb{T}^2} e^{2\pi i p(z-y)}e^{2\pi i p'(z'-y')}f(p,p',k)dpdp', \tag{153}$$

so we have $\mathfrak{D} J_{y,y'}(k) = \mathfrak{F}[i\delta\omega] J_{y,y'}(k)$, $\Theta J_{y,y'}(k) = \mathfrak{F}[\theta\omega] J_{y,y'}(k)$ etc. We can also write

$$i\big(\mathcal{K}_\xi^{(o)} \bar{G}^\epsilon\big)_{y,y'} = \mathfrak{F}\left[i\mathcal{I}_\xi r^{(o)}_{y,y}\right] \bar{G}^\epsilon_{y,y'}, \tag{154}$$

where

$$r^{(o)}_{y,y}(p,p',k,k') := r^{(+)}_{y,y'}(k+p',k') + r^{(-)}_{y,y'}(-k+p,k'), \tag{155}$$

and the stochastic integral $\mathcal{I}_\xi$ is performed with respect to the variable k'. Assume that, for some functions $f_+(k,k',p,p')$ and $f_-(k,k',p,p')$, $\bar{G}^{1,\epsilon}_0$ is given by

$$\big(\bar{G}^{1,\epsilon}_0(\xi)\big)_{y,y'} = i\mathfrak{F}\left[\mathcal{I}_\xi f_+ r^{(+)}_{y,y'}\right] \bar{G}^\epsilon_{y,y} + i\mathfrak{F}\left[\mathcal{I}_\xi f_- r^{(-)}_{y,y'}\right] \bar{G}^\epsilon_{y,y'}, \tag{156}$$

where $r^{(+)}$, $r^{(-)}$ depend on variables p, p', k, k' as on the right hand side of (155). Then, according to the left hand side of (152), we have

$$\mathfrak{Q}\big(\bar{G}^{1,\epsilon}_0(\xi)\big)_{y,y'}(k) = -\mathfrak{F}\left[i\mathcal{I}_\xi \gamma f_+ r^{(+)}_{y,y'}\right] \bar{G}^\epsilon_{y,y}(k) - \mathfrak{F}\left[i\mathcal{I}_\xi \gamma f_- r^{(-)}_{y,y'}\right] \bar{G}^\epsilon_{y,y}(k), \tag{157}$$

where γf stands for the function $\gamma(k') f(p,p',k,k')$. On the other hand

$$\mathfrak{D}\big(\bar{G}^{1,\epsilon}_0(\xi)\big)_{y,y'}(k) = \mathfrak{D}\mathfrak{F}\left[i\mathcal{I}_\xi f_+ r^{(+)}_{y,y'}\right] \bar{G}^\epsilon_{y,y}(k) + \mathfrak{D}\mathfrak{F}\left[i\mathcal{I}_\xi f_- r^{(-)}_{y,y'}\right] \bar{G}^\epsilon_{y,y}(k)$$

and we expand

$$\mathfrak{D}\mathfrak{F}\left[i\mathcal{I}_\xi f_+ r^{(+)}_{y,y'}\right] \bar{G}^\epsilon_{y,y}(k) =$$
$$\sum_{z,z'} \mathfrak{F}\left[i\mathcal{I}_\xi f_+ r^{(+)}_{z,z'}\right] \bar{G}^\epsilon_{z,z'}(k) \int_{\mathbb{T}^2} \mathrm{e}^{2\pi i p(z-y)} \mathrm{e}^{2\pi i p'(z'-y')} i\delta\omega(p,p',k)\, dp\, dp'.$$

Further expanding shows that this expression is equal to

$$\sum_{z,z'} \sum_{w,w'} G^\epsilon_{w,w'}(k) \int_{\mathbb{T}^2} \mathrm{e}^{2\pi i q(w-z)} \mathrm{e}^{2\pi i q'(w'-z')} i\mathcal{I}_\xi f_+ r^{(+)}_{z,z'}(q,q',k)\, dq\, dq'$$
$$\times \int_{\mathbb{T}^2} \mathrm{e}^{2\pi i p(z-y)} \mathrm{e}^{2\pi i p'(z'-y')} i\delta\omega(p,p',k)\, dp\, dp', \tag{158}$$

wherein

$$\mathcal{I}_\xi f_+ r^{(+)}_{z,z'}(q,q',k) = \int_{\mathbb{T}} f_+(q,q',k,k') r\left(k+q',k'\right) \mathrm{e}^{2\pi i k' z'} \widehat{\xi}(dk').$$

After changing the order of integration and grouping together factors depending on z, z', Dirac measures on $\mathbb{T}$ appear under the integrals in (158)

$$\delta(p-q) = \sum_z \mathrm{e}^{2\pi i(p-q)z}, \qquad \delta(p'+k'-q') = \sum_{z'} \mathrm{e}^{2\pi i(p'+k'-q')z'},$$

so (158) reduces to

$$i\sum_{w,w'} G^{\epsilon}_{w,w'}(k)\int_{\mathbb{T}^3} \mathrm{e}^{2\pi i(pw+p'w')}\, f_+(p,p'+k',k,k')r(k+p'+k',k')e^{2\pi ik'w'}\,\widehat{\xi}(dk') \\ \times \mathrm{e}^{-2\pi i(py+p'y')} i\delta\omega(p,p',k)dpdp'.$$

Eventually we get

$$\mathfrak{D}\mathfrak{F}\left[i\mathcal{I}_\xi f_+ r^{(+)}_{y,y'}\right]\bar{G}^{\epsilon}_{y,y}(k) = i\sum_{w,w'} G^{\epsilon}_{w,w'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi ip(w-y)}\mathrm{e}^{2\pi ip'(w'-y')} \\ \times \int_{\mathbb{T}} i\delta\omega(p,p'-k',k)f_+(p,p',k,k')r(k+p',k')\mathrm{e}^{2\pi ik'y'}\,\widehat{\xi}(dk')dpdp'.$$

Similarly we obtain

$$\mathfrak{D}\mathfrak{F}\left[i\mathcal{I}_\xi f_- r^{(-)}_{y,y'}\right]\bar{G}^{\epsilon}_{y,y}(k) = i\sum_{w,w'} G^{\epsilon}_{w,w'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi ip(w-y)}\mathrm{e}^{2\pi ip'(w'-y')} \\ \times \int_{\mathbb{T}} i\delta\omega(p-k',p',k)f_-(p,p',k,k')r(-k+p',k')\mathrm{e}^{2\pi ik'y}\,\widehat{\xi}(dk')dpdp',$$

Recalling also (154) and (157) we get that $\bar{G}^{1,\epsilon}_0$ is given by (156) with

$$f_+ := \Big[\gamma(k') + i\big(\omega(k-k'+p') - \omega(k-p)\big)\Big]^{-1},$$
$$f_- := \Big[\gamma(k') + i\big(\omega(k+p') - \omega(k+k'-p)\big)\Big]^{-1}.$$

Formulas for $\bar{G}^{1,\epsilon}_+$ and $\bar{G}^{1,\epsilon}_-$ are

$$\big(\bar{G}^{1,\epsilon}_+(\xi)\big)_{y,y'} = -i\mathfrak{F}\left[\mathcal{I}_\xi g_+ r^{(+)}_{y,y'}\right]\bar{G}^{\epsilon}_{y,y}, \qquad \big(\bar{G}^{1,\epsilon}_-(\xi)\big)_{y,y'} = -i\mathfrak{F}\left[\mathcal{I}_\xi g_- r^{(-)}_{y,y'}\right]\bar{G}^{\epsilon}_{y,y},$$

where

$$g_+ := \Big[\gamma(k') - i\big(\omega(k-k'+p') + \omega(k-p)\big)\Big]^{-1},$$
$$g_- := \Big[\gamma(k') + i\big(\omega(k+p') + \omega(k+k'-p)\big)\Big]^{-1}.$$

By $\mathfrak{G}_2^\epsilon(t), \mathfrak{G}_{2,+}^\epsilon(t), \mathfrak{G}_{2,-}^\epsilon(t)$ in the third line of (151) we denote

$$\mathfrak{G}_2^\epsilon(t) := i\widetilde{\mathcal{K}}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{1,\epsilon}(t) - i\widetilde{\mathcal{K}}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}_{+,\mathrm{e}}^{1,\epsilon}(t) - i\widetilde{\mathcal{K}}^{(-)}_{\xi^{(\epsilon)}(t)}\bar{G}_{-,\mathrm{e}}^{1,\epsilon}(t) + (\mathfrak{Q} - \mathfrak{D})\bar{G}_0^{2,\epsilon}(t),$$
$$\mathfrak{G}_{2,+}^\epsilon(t) := i\widetilde{\mathcal{K}}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}_+^{1,\epsilon}(t) - i\widetilde{\mathcal{K}}^{(+)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{1,\epsilon}(t) + (\mathfrak{Q} + i\Theta)\bar{G}_+^{2,\epsilon}(t), \tag{159}$$
$$\mathfrak{G}_{2,-}^\epsilon(t) := i\widetilde{\mathcal{K}}^{(o)}_{\xi^{(\epsilon)}(t)}\bar{G}_-^{1,\epsilon}(t) - i\widetilde{\mathcal{K}}^{(-)}_{\xi^{(\epsilon)}(t)}\bar{G}_0^{1,\epsilon}(t) + (\mathfrak{Q} - i\Theta)\bar{G}_-^{2,\epsilon}(t).$$

$\bar{G}_0^{2,\epsilon}(t), \bar{G}_+^{2,\epsilon}(t), \bar{G}_-^{2,\epsilon}(t)$ are defined as random fields such that

$$\mathfrak{G}_2^\epsilon(t) = \mathfrak{G}_2^\epsilon(t) = \mathfrak{G}_2^\epsilon(t) = 0$$

μ_σ–almost surely. The solutions are given by (153) with kernels f (which depend also on variables y and y') being double stochastic integrals. We do not write down lengthy formulas for the solutions, we only note that they do not contribute to the limiting scattering operator. They appear in the last line of (151) which vanishes. Now let us display components abbreviated by $\mathfrak{G}^\epsilon, \mathfrak{G}_+^\epsilon, \mathfrak{G}_-^\epsilon$ in the first line of (151). They are

$$\mathfrak{G}^\epsilon = (\mathcal{L}_\epsilon^{(o)} - \mathfrak{D}_\epsilon)\bar{G}^\epsilon,$$
$$\mathfrak{G}_+^\epsilon = \mathcal{L}_\epsilon^{(+)}\bar{G}^\epsilon, \quad \mathfrak{G}_-^\epsilon = \mathcal{L}_\epsilon^{(-)}\bar{G}^\epsilon,$$

where

$$\mathcal{L}_\epsilon^{(o)}\bar{G}^\epsilon = \int_{H_\lambda}\Big(i\mathcal{K}_\xi^{(o)}\bar{G}_0^{1,\epsilon}(\xi) - i\mathcal{K}_\xi^{(+)}\bar{G}_{+,\mathrm{e}}^{1,\epsilon}(\xi) - i\mathcal{K}_\xi^{(-)}\bar{G}_{-,\mathrm{e}}^{1,\epsilon}(\xi)\Big)\mu_\sigma(d\xi), \tag{160}$$
$$\mathcal{L}_\epsilon^{(+)}\bar{G}^\epsilon = \int_{H_\lambda}\Big(i\mathcal{K}_\xi^{(o)}\bar{G}_+^{1,\epsilon}(\xi) - i\mathcal{K}_\xi^{(+)}\bar{G}_0^{1,\epsilon}(\xi)\Big)\mu_\sigma(d\xi),$$

and

$$\mathcal{L}_\epsilon^{(-)}\bar{G}^\epsilon = \int_{H_\lambda}\Big(i\mathcal{K}_\xi^{(o)}\bar{G}_-^{1,\epsilon}(\xi) - i\mathcal{K}_\xi^{(-)}\bar{G}_0^{1,\epsilon}(\xi)\Big)\mu_\sigma(d\xi).$$

Deriving more explicit formulas for $\mathcal{L}_\epsilon^{(\cdot)}$ demands some straightforward although lengthy calculations. Let us take a look at

$$\begin{aligned}\left[i\mathcal{K}_\xi^{(o)}\bar{G}_0^{1,\epsilon}(\xi)\right]_{y,y'}(k) &= \mathfrak{F}\left[i\mathcal{I}_\xi r_{y,y'}^{(o)}\right]\left(\bar{G}_0^{1,\epsilon}(\xi)\right)_{y,y'}(k)\\ &= \mathfrak{F}\left[i\mathcal{I}_\xi r_{y,y'}^{(o)}\right]\left(i\mathfrak{F}\left[\mathcal{I}_\xi f_+ r_{y,y'}^{(+)}\right] + i\mathfrak{F}\left[\mathcal{I}_\xi f_- r_{y,y'}^{(-)}\right]\right)\bar{G}_{y,y}^{\epsilon}\\ &= -\sum_{\iota,\iota'\in\{+,-\}}\mathfrak{F}\left[\mathcal{I}_\xi r_{y,y'}^{(\iota)}\right]\mathfrak{F}\left[\mathcal{I}_\xi f_{\iota'} r_{y,y'}^{(\iota')}\right]\bar{G}_{y,y}^{\epsilon}. \qquad (161)\end{aligned}$$

We expand

$$\mathfrak{F}\left[\mathcal{I}_\xi r_{y,y'}^{(+)}\right]\mathfrak{F}\left[\mathcal{I}_\xi f_+ r_{y,y'}^{(+)}\right]\bar{G}_{y,y'}^{\epsilon}(k) =$$

$$\sum_{z,z'}\mathfrak{F}\left[\mathcal{I}_\xi f_+\, r_{z,z'}^{(+)}\right]\bar{G}_{z,z'}^{\epsilon}(k)\int_{\mathbb{T}^2}\mathrm{e}^{2\pi i p(z-y)}\mathrm{e}^{2\pi i p'(z'-y')}\mathcal{I}_\xi r_{y,y'}^{(+)}(k+p')dpdp' =$$

$$\begin{aligned}&= \sum_{w,w'}\bar{G}_{w,w'}^{\epsilon}(k)\int_{\mathbb{T}^2}\mathrm{e}^{2\pi i p w}\mathrm{e}^{2\pi i p' w'}\int_{\mathbb{T}}f_+(p,p'+l,k,l)r(k+p'+l,l)\mathrm{e}^{2\pi i l w'}\,\widehat{\xi}(dl)\\ &\quad\times\mathrm{e}^{-2\pi i p y}\mathrm{e}^{-2\pi i p' y'}\int_{\mathbb{T}}r(k+p',k')\mathrm{e}^{2\pi i k' y'}\,\widehat{\xi}(dk')dpdp'.\end{aligned}$$

We integrate this with respect to the measure μ_σ, and since

$$\int\widehat{\xi}(dk)\widehat{\xi}(dk')\mu_\sigma(d\xi) = \delta(k+k')\sigma(k)dk,$$

we obtain

$$\begin{aligned}&\int\mathfrak{F}\left[\mathcal{I}_\xi r_{y,y'}^{(+)}\right]\mathfrak{F}\left[\mathcal{I}_\xi f_+ r_{y,y'}^{(+)}\right]\bar{G}_{y,y'}^{\epsilon}(k)\,\mu_\sigma(d\xi) =\\ &\quad\sum_{w,w'}\bar{G}_{w,w'}^{\epsilon}(k)\int_{\mathbb{T}^2}e^{2\pi i p(w-y)}e^{2\pi i p'(w'-y')}\times\\ &\quad\quad\int_{\mathbb{T}}\frac{[r(k+p',k-k')]^2\sigma(k-k')}{\gamma(k-k')+i\big(\omega(k'+p')-\omega(k-p)\big)}\,dk'dpdp'.\end{aligned}$$

We have also used here symmetries of functions involved, in particular the property that $r(k+l,l) = r(k,-l)$ for all $k,l \in \mathbb{T}$, and some change of variables. Now by inserting test function $G_{\epsilon,w,w}(k) := G(\epsilon(w+w')/2,k)$, $G(\cdot,\cdot) \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$, into the above formula, we get

$$\widetilde{\mathcal{L}}_1 G_{\epsilon,y,y'}(k) := -\int \mathfrak{F}\left[\mathcal{I}_\xi r^{(+)}_{y,y'}\right] \mathfrak{F}\left[\mathcal{I}_\xi f_+ r^{(+)}_{y,y'}\right] G_{\epsilon,y,y'}(k)\,\mu_\sigma(d\xi) =$$

$$\sum_z G(\epsilon z/2,k)\int_{\mathbb{T}} e^{2\pi i p(z-y-y')}\int_{\mathbb{T}} \widetilde{R}(p,k,k')dk'dp,$$

where

$$\widetilde{R}(p,k,k') := \frac{-[r(k+p,k-k')]^2\sigma(k-k')}{\gamma(k-k')+i\big(\omega(k'+p)-\omega(k-p)\big)}.$$

Pairing $\widetilde{\mathcal{L}}_1 G_\epsilon$ with $\mathcal{W}^\epsilon$ expands as follows

$$\langle \mathcal{W}^\epsilon, \widetilde{\mathcal{L}}_1 G_\epsilon\rangle =$$

$$= \sum_{y,y'}\int_{\mathbb{T}} \mathcal{W}^\epsilon_{y,y'}(k)\sum_z G^*(\epsilon z/2,k)\int_{\mathbb{T}} e^{2\pi i p(y+y'-z)}\int_{\mathbb{T}} \widetilde{R}^*(p,k,k')\,dk'dp\,dk$$

$$= \frac{\epsilon}{2}\Big\langle \sum_{z,y,y'\in\mathbb{Z}}\int_{\mathbb{T}} \psi^{(\epsilon)}_y(t)\left[\psi^{(\epsilon)}_{y'}(t)\right]^* e^{2\pi i k(y'-y)}\,G^*(\epsilon z/2,k)\times$$

$$\times\int_{\mathbb{T}} e^{2\pi i p(y+y'-z)}\int_{\mathbb{T}} \widetilde{R}^*(p,k,k')dk'dp\,dk\Big\rangle_{\mu_\epsilon}$$

$$= \int_{-1/(2\epsilon)}^{1/(2\epsilon)}\int_{\mathbb{T}} \frac{\epsilon}{2}\Big\langle \widehat{\psi}^{(\epsilon)}\left(t,k+\frac{\epsilon p}{2}\right)\widehat{\psi}^{(\epsilon)}\left(t,k-\frac{\epsilon p}{2}\right)\Big\rangle_{\mu_\epsilon}\times$$

$$\frac{\epsilon}{2}\sum_{z\in\mathbb{Z}} G^*(\epsilon z/2,k)e^{\pi i\epsilon p z}\int_{\mathbb{T}} \widetilde{R}^*(-\epsilon p,k,k')dk'dp\,dk.$$

In the limit, the product of function $G(p,k)$ and the factor $\int_{\mathbb{T}} \widetilde{R}(0,k,k')\,dk'$ is one of the building components of scattering operator – generator of compound Poisson process on $\mathbb{T}$. Another component of (161) expands as follows

$$\mathfrak{F}\left[\mathcal{I}_\xi r^{(-)}_{y,y'}\right]\mathfrak{F}\left[\mathcal{I}_\xi f_- r^{(-)}_{y,y'}\right]\bar{G}^{\epsilon}_{y,y'}(k) =$$

$$= \sum_{w,w'} \bar{G}^{\epsilon}_{w,w'}(k)\int_{\mathbb{T}^2} \mathrm{e}^{2\pi i p w}\mathrm{e}^{2\pi i p' w'}\int_{\mathbb{T}} f_-(p+l,p',k,l) r(-k+p+l,l)\mathrm{e}^{2\pi i l w}\,\widehat{\xi}(dl)$$

$$\times\, \mathrm{e}^{-2\pi i p y}\mathrm{e}^{-2\pi i p' y'}\int_{\mathbb{T}} r(-k+p,k')\mathrm{e}^{2\pi i k' y}\,\widehat{\xi}(dk')dp dp'.$$

After integrating and inserting G_ϵ we obtain

$$\widetilde{\mathcal{L}}_2 G_{\epsilon,y,y'}(k) := -\int \mathfrak{F}\left[\mathcal{I}_\xi r^{(-)}_{y,y'}\right]\mathfrak{F}\left[\mathcal{I}_\xi f_- r^{(-)}_{y,y'}\right] G_{\epsilon,y,y'}(k)\,\mu_\sigma(d\xi) =$$

$$\sum_z G(\epsilon z/2,k)\int_{\mathbb{T}} e^{2\pi i p(z-y-y')}\int_{\mathbb{T}} \widetilde{R}^*(p,k,k')\,dk' dp.$$

We have

$$\widetilde{R}(0,k,k') + \widetilde{R}^*(0,k,k') = \frac{-2r^2(k,k-k')\gamma(k-k')\sigma(k-k')}{\gamma^2(k-k') + \big(\omega(k')-\omega(k)\big)^2}$$

which is a part of the kernel of scattering generator. The remaining components of (160) give rise to remaining parts of the operator.

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