POLITECHNIKA LUBELSKA

Zbigniew Łagodowski

Convergence of random fields

Selected methods

MONOGRAFIE

Lublin 2021

Convergence of random fields

Selected methods

Monografie – Politechnika Lubelska

Politechnika Lubelska
Wydział Elektrotechniki i Informatyki
ul. Nadbystrzycka 38A
20-618 Lublin

Zbigniew Łagodowski

Convergence of random fields

Selected methods

Wydawnictwo
Politechniki Lubelskiej

Lublin 2021

Recenzent:
dr hab. Przemysław Matula, prof. UMCS w Lublinie

Publication approved by the Rector of Lublin University of Technology

ISBN: 978-83-7947-497-4

Publisher: Wydawnictwo Politechniki Lubelskiej
www.biblioteka.pollub.pl/wydawnictwa
ul. Nadbystrzycka 36C, 20-618 Lublin
tel. (81) 538-46-59

Contents

Introduction

The natural development of limit theorems in probability theory has brought the wealth of generalizations of classical results. Diversity ideas and results don't allow to study all that problems within the framework of a single monograph. We will focus on a "territory" of limit theorems, if the indexes of the random variables are taken from a more general space than $\mathbb{N}$ or $\mathbb{R}$. Thus we study limit theorems for random variables with indices include numerous discrete or continuous coordinates. The set of multiple indexes we deal with is denoted by $\mathbb{N}^d$ or $\mathbb{R}^d$, where d is the given natural number. Any family of random variables indexed by such spaces is called a random field.

In the introduction to the fundamental monograph [117], John B. Walsh wrote that random field research in itself is very interesting. However, one can point out many reasons, not only purely mathematical ones, that justify the need to study laws regarding random variables with a multi-parameter index. The nature of natural phenomena that we want to describe is usually multidimensional. Random fields appear in the analysis of biological and medical data, in physics and statistical mechanics and in geophysics, in particular in seismography and astrophysics. Examples of such research can be found in the publications [2], [19], [20], [18], [116], [122], [123] and [124], to name only the works showing especially interesting applications in the field of imaging the surface of the brain and its functioning.

Examples from other areas, however, are much more numerous. In statistics, the convergence of random variables with a multidimensional index is the basis for the construction of a number of non-parametric tests, for example the Mann-Whitney-Wilcoxon test, the Wald-Wolfowitz series test and non-parametric analysis of variance – the Kruskal-Wallis test. The almost sure or complete convergence of random fields that I examined is applicable in the almost certain convergence of multi-sample U-statistics, in non-parametric regression models and in EV-type (Errors-in-Variables) linear models in which the errors are mutually dependent (cf. [43], [127]).

Let's go to brief discussion of the presented results. In this monograph we deal with theorems of the probability theory on limiting behavior of random fields with the following structures:

- independence (of random variables and random elements),
- negative association (NA),
- negative dependence (ND),
- asymptotic pairwise negative dependence (APND),
- martingale.

as well as theorems on limiting behavior of random fields with random multindex and sums of random elements taking values in Banach spaces. Most results refer to the latter structure; all were obtained for convergence in the max mode, while maintaining such moment assumptions as for convergence of random fields in the min mode. Let us now go to a more detailed presentation of the results.

In the second chapter we present Burkholder and Rosenthal inequalities for a martingale and negative associated random fields. The application of these inequalities made it possible to obtain Baum-Katz theorems for random fields of negative association and martingale structure, characterizing the rate of convergence in the Marcinkiewicz type laws of large numbers of the form

$$\frac{S_{\mathbf{n}}}{(n_1 \cdot n_2 \cdot \ldots \cdot n_d)^{\alpha}} = \frac{S_{\mathbf{n}}}{|\mathbf{n}|^{\alpha}} \to 0 \text{ a.s., as } \max \mathbf{n} \to \infty,$$

where $|\mathbf{n}| := n_1 \cdot n_2 \cdot \ldots \cdot n_d$.

In the third chapter we present the methods of proving the complete convergence based on Kahane-Hoffmann-Jørgensen inequality. This inequality has no equivalent for sequences of dependent random variables, much less for random fields with a dependence structure. The weaker inequality of Kahane-Hoffmann-Jørgensen, established by us for some random fields with a dependence structure, is still useful for the study of complete convergence. The result that allows us to realize this idea are the Fuk-Nagaev inequalities; we prove them for negatively dependent random fields and martingales. Thanks to these results, we obtain Kahane-Hoffmann-Jørgensen inequalities for random fields with the dependence structures listed above. This in turn allowed us to obtain Baum-Katz theorems characterizing the rate of convergence in Marcinkiewicz laws of large numbers of the form

$$\frac{S_n}{n_1^{\alpha_1} \cdot n_2^{\alpha_2} \cdot \ldots \cdot n_d^{\alpha_d}} \to 0 \text{ a.s., as } \max \mathbf{n} \to \infty.$$

In this section, we propose a general method of proof of the Baum-Katz theorems, for random fields with any dependence structure.

To locate the results of the second and third section, compare them to the existing literature in this respect. Robert Smythe, Galen Shorack, Alan Gut and Oleg Klesov are the authors of most of the results regarding almost sure convergence of random fields; but nearly all of them are achieved on the assumption of the same distribution and independence of random variables, the methods of proof not transferable into random fields with a structure of dependence and composed of random variables with different distributions. Baum-Katz theorems were basically exclusively developed by Gut and Klesov and their associates, Peligrad and Stadtmüller. The most general results were the statements contained in the works: [53], [104] and [68]. We extend the quoted results to random fields with different dependencies and not necessarily the same distribution. In this way, we contribute to the creation of a consistent and definitely more complete family of Baum-Katz theorems for random fields. The fourth chapter is substantively quite close to previous section. What are the differences between the results of chapter 3 and of this one? The Fuk-Nagaev inequality presented in the fourth section is valid for martingale and as well as for reversed martingale for conditional moments $r \geq 2$ while the tail probabilities are determined by the real field $\{y_{\mathbf{k}} > 0,\ \mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}\}$.

In the fifth chapter, based on the results of [81], we generalize very interesting weighted strong law of large numbers, obtained by Jajte in [59]. Due to the large freedom of selection of weighting and normalizing functions, the result can be seen as almost sure convergence of (h, g) transforms of random field or the (h, g) method of their summability. Included there are such methods of summability as: Cesàro $(C, 1)$ mean, logarithmic means, transforms of Marcinkiewicz strong law of large numbers types, etc.

In the next part of this chapter, we prove Kolmogorov SLLN for random fields with a dependence structure in pairs, defined on the basis of the concept of copulas. The structure thus defined contains random fields (sequences of random variables) negatively dependent in pairs (quadrant-wise, in particular pairwise independent) as well as structures defined by Farlie-Gumbel-Morgerstern and Ali-Mikhail-Haq copulas as well as the Plackett copulas family. The random fields considered are related to fields of asymptotically quadrant independent and asymptotically quadrant sub-independent random variables. In this chapter, we also provide the missing version, proven in [81], of the second Borel-Cantelli lemma, for events dependent in pairs, with the structure of dependence the same as the above considered random field; a result necessary in the proof of the strong laws of large numbers mentioned. All the results of this chapter were obtained for convergence in the max mode and parallel for sectoral convergence.

In the sixth and seventh chapter we discuss methods of proving the strong of large numbers for random fields taking value in Banach spaces. Thus, the sixthth chapter is devoted mainly to the presentation of some evidence techniques in the absence of Doob inequality for martingale random fields. A partial solution of the problem is achieved by generalizing the quasi-inequalities of Hajek-Réni-Chow, demonstrated by Serfling and Christofides in [23] (cf. Theorem 2.1, p. 633). Maintaining the assumptions of the previously quoted theorem of Serfling and Christofides (SC theorem), we modified the event whose probability we estimated, which gave us the tool to study almost sure convergence of random fields in the max mode (theorem SC only allows for convergence in the min mode). The extension of the Serfling-Christofides inequality to random fields with sub-martingale structure allowed to apply it to almost sure convergence of random fields with values in the Banach space. In the second part of this chapter we present Marcinkiewicz inequality for fields of random elements. This result allowed to receive the Brunk-Prokhorov type of SLLN for random elements taking values in Banach space. In the next chapter we show that in order to obtain the Feller's strong law of large numbers it is enough that the sum of random elements indexed with certain subsets of $\mathbb{N}^d$ satisfies the asymptotic condition (this is also a necessary condition). We don't assume any codnition about the geometry of Banach space $(\mathbb{B}, \| \|)$.

In the eighth chapter we deal with the weak convergence of fields of random elements in the metric space. More precisely, in the first section we give the sufficient conditions, and in most cases also necessary, for the random field $\{Y_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ weakly convergent to the certain measure μ, to ensure the weak convergence of $\{Y_{N_{\mathbf{n}}},\, \mathbf{n} \in \mathbb{N}^d\}$ to the same measure, without imposing any conditions on the probabilistic relations between random fields and field of random indices.

In the last chapter we provide the results giving the rate of convergence in the weak law of large numbers for fields of independent random variables and martingales, for their partial sums indexed randomly.

This monograph is based on my unpublished selfreview scientific achievements, prepared as the habilitation thesis in mathematical sciences, successfully achieved in 2020. Additionally we give some proofs of most important theorems.

Chapter 1

Random field convergence problems

1.1. Historical outline

It appears that the first person studying the problem of convergences of families of random variables with multidimensional indices was Norbert Wiener who, in 1939, in his work „The ergodic theorem" [119] examined the convergence of the sum of d parameter dependent measurable functions. More specifically, he considered the measurable mapping of T in space with measure $(\Omega, \mathfrak{F}, \mu)$ preserving measure μ and studied limit properties of sums of type:

$$S_m[f] = \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_d=1}^{m} f \circ T^{k_1+k_2+\cdots+k_d}.$$

He proved that for $d \geq 1$ and $f \in L^1(\Omega, \mathfrak{F}, \mu)$ the limit

$$\lim_{m\to\infty} \frac{S_m[f]}{m^d} \tag{1}$$

exists for almost all $\omega \in \Omega$ with respect to μ. Because of the fact that the partial sums considered by Wiener had such a property that each of them contained all previous ones, conditions for convergence turned out to be the same as in the case of the classic ergodic theory. Several years later, independently of each other, Dunford in [35] and Zygmund in [130] endowed this problem with a truly multidimensional sense, assuming

$$S_{\mathbf{n}}[f] = \sum_{\mathbf{k} \preceq \mathbf{n}} f \circ T^{k_1+k_2+\cdots+k_d} \quad \text{dla} \quad \mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d, \tag{2}$$

where $\mathbf{k} \preceq \mathbf{n}$ means that $k_i \leq n_i$ for each i from the set $\{1, 2, \ldots, d\}$; these sums no longer have the properties referenced above, and the problem of their convergence cannot be brought down to the convergence of "ordinary" sums. Dunford and Zygmund proved that the limit

$$\lim_{\min \mathbf{n} \to \infty} \frac{S_{\mathbf{n}}[f]}{n_1 \cdot n_2 \cdot \ldots \cdot n_d} \tag{3}$$

($\min \mathbf{n}$ is the smallest of the coordinates of vectors $\mathbf{n} = (n_1, \ldots, n_d)$) exists almost everywhere with respect to measure μ, as long as the function f satisfies the condition

$$\int_{\Omega} |f(\omega)| \left(\log^+ |f(\omega)|\right)^{d-1} d\mu < \infty, \tag{4}$$

where $\log^+ x = \max(0, \log x)$. In a special case where μ is a probabilistic measure, and the function f is a random variable X, for which there is a moment

$$\mathbb{E}\, |X| \left(\log^+ |X|\right)^{d-1} < \infty \tag{5}$$

from the result of Wiener, Dunford and Zygmund we obtain the Strong Law of Large Numbers (SLLN) for weakly stationary random field $X_{\mathbf{k}} = T^{k_1+k_2+,\ldots,+k_d} \circ X$, which refers to the existence of an almost sure limit

$$\lim_{\min \mathbf{n} \to \infty} \frac{\sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}}{|\mathbf{n}|} = \mathbb{E}\, X. \tag{6}$$

As Smythe showed in [111], in the case of fields of independent random variables with the same distribution, condition (5) is also necessary for the strong law of large numbers (6). Another important step was made by Klaus Krickeberg, when in the mid

1950s he began to examine the convergence of countably additive collection functions, which in essence are martingales indexed by directed sets.To be precise, it should be added that chronologically, Krieckeberg was overtaken by Bochner, who in [11] introduced the notion of such a martingale.

Significant progress in the study of the convergence of independent random variables and martingales indexed by partially ordered sets was noted in the 1970s, that is 30 years after the above-mentioned work of Wiener. At the beginning of the 1980s, research on other types of dependencies of random variables with multidimensional indexes and their limiting properties began. The authors of the most important works include Wichura [118], Cairoli [16], Walsh [117], Chaterji [21] and [22], Smythe, Shorack [111] and [110], Millet, Sucheston [97] and [98], Wong and Zakai, [120], [126], [121], Merzbach [92], [93] and [94], Gut [48], [50] i [49], Klesov [63], [64], [65] and [66]. Research continues.

1.2. Specificity of random field studies

Modes of multiindex divergence

Limit theorems concerning random fields, i.e. families of random variables indexed with elements $\mathbb{N}^d$, usually designated

$$\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\},$$

can be distinguished depending on the *mode* in which the index $\mathbf{n}$ tends to infinity and the rule for selecting the "vertex of the rectangle" determining partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}$. For example, all the "vertices" $\mathbf{n}$, in the sums that Wiener considered, were on the "diagonal" (on which all index coordinates are the same); such a special selection of sums meant that the theorems he obtained could be proved by the methods used for random variable sequences. The convergence mode that Dunford and Zygmund considered was already demanding new tools; recall (see (3)) that in their works the indices $\mathbf{n}$, of partial sums $S_{\mathbf{n}}$, were arbitrary and tended to infinity in the sense that $\min \mathbf{n}$ tends to infinity ; this is the most common mode in the literature to this day. Such convergence will be called mode of min convergence.

The second most commonly used convergence mode is convergence when $\max \mathbf{n}$ (the largest of the coordinates of the vector $\mathbf{n} = (n_1, \dots, n_d)$) tends to infinity; we will call it convergence in the max mode. This mode of convergence appears in all the results presented in this outline; it is obviously stronger than convergence in the min mode. In some works convergence in the max mode is called strong convergence, in contrast to the most commonly used convergence in the min mode, considered as a standard and designated $\mathbf{n} \to \infty$ (rather historically, but sometimes also called "convergence in Pringsheim's sense", cf. [99]). We accept the convention introduced by Klesov in his monograph [68] and will use the terms max mode and min mode of convergence. There are of course other definitions of the convergence mode, but they are practically not considered at all.

We also deal with a very important type of convergence that does not have the characteristics of any convergence mode. Namely, we study the limiting behavior of partial sums $S_{\mathbf{n}}$, whose indices $\mathbf{n}$ belong to the infinite set $A \subset \mathbb{N}^d$. We will call this type of convergence a sectoral convergence. It has the interesting feature that the classical assumptions, as in a one-dimensional case, are sufficient for the strong laws of large numbers. This type of convergence will be discussed in detail in chapter five.

Types of convergence

As is known, certain types of convergence of random variables, including weak convergence and convergence in probability, can be metrizable, and others, such as almost sure and complete convergence – cannot. In the theory of random fields, claims about this second class of convergence are definitely more challenging, since we can not apply the following well-known lemma, which provides a sufficient condition for the convergence of generalized sequences.

Lemma 1.2.1 ([102], Lemma V-1-1)**.** *In order that family $\{x_t, t \in T\}$, indexed by directed set T, converge to x in the complete metric space S , it suffices $\{x_{t_n}, n \in \mathbb{N}\}$ be convergent to x for all increasing sequences $\{\mathbf{t}_n, n \in \mathbb{N}\}$ in T.*

If we limit ourselves to the case which is the subject of our considerations, i.e. when $T = \mathbb{N}^d$; then, for both the max and min mode of convergence we can specify the necessary and sufficient conditions for the convergence of elements of any metric space (see [83], Lemma 2.2 or [68], Corollary A.1 and A.2). Thanks to this, some issues concerning the convergence of random fields for metrizable convergence can be reduced to the study of the convergence of subfields or sequences, and then use standard tools.

However, this path cannot be followed in the case of proofs of almost certain or complete convergence; this case requires new tools, new methods and – due to the fact that random fields are more complex than sequences – additional assumptions. In this outline we will only deal with non-matrizable convergences: almost sure and complete ones. However, it should be emphasized that some problems are common to both types of convergence; this results for example from the fact that convergences in the max and min mode are not equivalent; fields being norming families or weights for different types of convergence are generally not a simple generalization of sequences.

Problems of convergence of martingale random fields

The modern theory of stochastic processes is based on the theory of martingales, which in the last decades has been developed to such an extent that it has become an extremely important research tool. Let us start with the definition. Assume that the σ-field $\{\mathfrak{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a filtration on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, that is, the following condition is fulfilled

(F1) if $\mathbf{k} \preceq \mathbf{n}$ then $\mathfrak{F}_{\mathbf{k}} \subset \mathfrak{F}_{\mathbf{n}} \subset \mathfrak{F}$.

(In literature, this condition is historically marked (F1), as above, similarly (F2) and (F4), cf. [17], p. 113; we will stick to this convention). The integrable family of random variables $\{Z_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ adapted to the filtration $\{\mathfrak{F}_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ is called a martingale random field (or just a martingale) if

$$\bigwedge_{\mathbf{k} \preceq \mathbf{n}} \mathbb{E}\left(Z_{\mathbf{n}} | \mathfrak{F}_{\mathbf{k}}\right) = Z_{\mathbf{k}} \quad \text{a.s.}$$

By analogy, we define the concepts of super- and submartingale.

Unfortunately, it turns out that so defined martingales with multidimensional indices differ significantly from the classical ones. From the technical point of view, attempts to demonstrate their properties are often thwarted by the lack of a tool, which is Markov's moments, and in particular the moments of entry into a given set for the first time – due to the fact that the set of indices $\mathbb{N}^d$ is not linearly ordered, "time of first entry" it is not well-defined. But this lack is the result of something more fundamental: martingale random fields are much less regular than their classic counterparts (natural numbers indexed martingales). This is evidenced, by the example constructed by Cairoli (see [16]), that the analogue of Doob's classic inequality

$$\mathbb{P}(\max_{k \leq n} |M_k\ | \geq \lambda) \leq \lambda \mathbb{E}\,|M_n|,$$

where $\{M_n, n \in \mathbb{N}\}$ is a martingale with natural filtration – it is not true for martingale random fields.

Dubins and Pitman in 1980 (see [34]) found also a counter-example that for martinagles, as defined above, there is no equivalent to the statement, that

$$\sup_{n \geq 1} \mathbb{E}\,|M_n| < \infty$$

entails the existence of an almost sure limit of $(M_n)_{n \geq 1}$. Previously, an analogous example for a martingale indexed by directed set was given by J. Dieudonné in [32]. K. Krickeberg showed (see [70]) that for each bounded martingale indexed by directed set to be convergent (but only in the sense of essential convergence), it is necessary for the considered filtration to meet the so-called Vitali topological condition.

Very interesting results regarding the characterization of essential convergence of martingales were obtained by A. Millet, L. Sucheston [97], [98] and K. Astbury [6], who gave necessary and sufficient conditions for essential convergence, in the language of classic stopping time and multi-valued stopping time. These theorems correspond with the theory of Burkholder [15] and Chaterji [21], [22], combining almost sure convergence with almost sure maximal inequalities for martingales with discrete time. A summary of these results can be found in the monographs of Edgar and Sucheston [36].

In the case of martingales indexed by elements of space $\mathbb{N}^d$, a satisfying theory can be built with the assumption of conditional independence of filtration (CI condition), which can be defined as follows (see [62], p. 35):

(F4) $$\mathbb{E}\left(\mathbb{E}\left(\cdot|\mathfrak{F}_{\mathbf{m}}\right)|\mathfrak{F}_{\mathbf{n}}\right) = \mathbb{E}\left(\cdot|\mathfrak{F}_{\mathbf{m}\wedge\mathbf{n}}\right) \quad \text{a.s.}$$

Let us add that partial sums of independent random variables with a multidimensional indices form a martingale random field with respect to natural filtration, which in this case fulfills condition (F4).

Chapter 2

Maximal inequalities for moments and Baum-Katz type theorems

This section is devoted to Rosenthal and Burkholder inequalities, proven in [84], for random fields with different dependencies, and with their help, the Baum-Katz type theorems were obtained, which give information about the rate of convergence in Marcinkiewicz-Zygmund laws of large numbers, as well as about complete convergence in these laws.

2.1. Results for negatively associated random fields

Let us begin with a reminder of the basic definition and auxiliary notations. Let $(\mathbf{n}) := \{\mathbf{k} \in \mathbb{N}^d : \ \mathbf{k} \preceq \mathbf{n}\}$.

Definition 2.1.1. *A finite family of random variables* $\{X_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n}\}$ *is said to be negatively associated if for every pair of disjoint subset* S, T *of* $(\mathbf{n})$ *and any pair of coordinate-wise nondecreasing functions*

$$f : \mathbb{R}^{|S|} \to \mathbb{R}, \qquad g : \mathbb{R}^{|T|} \to \mathbb{R},$$

it follows that

$$\mathbf{cov}(f(X_{\mathbf{j}}, \mathbf{j} \in S), g(X_{\mathbf{k}}, \mathbf{k} \in T)) \le 0,$$

provided the covariance exists.

We say that random field $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ is negatively associated, if for each $\mathbf{n} \in \mathbb{N}^d$ subfamily $\{X_{\mathbf{k}},\ \mathbf{k} \preceq \mathbf{n}\}$ is negatively associated.

Random variables with negatively associated structure naturally appear, for example, as results of drawing from a finite population without replacement. Many known multidimensional distributions, including multidimensional hyper-geometric distribution, negatively correlated normal distribution, polynomial distribution and multivariate Dirichlet distribution, describe fields or sequences of negatively associated random variables (see Joag-Dev and Proschan [61]).

One of the basic tools that allows to prove the Baum-Katz limit theorems is Rosenthal inequality. The following Lemma is a version of Zhang and Wen result (see [128], lemma A.2) transformed into a form with the attributes of the above-mentioned inequality.

Lemma 2.1.2. ([84], Lemma 2.1) *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a negatively associated random field with*

$$\mathbb{E}\, X_{\mathbf{n}} = 0 \quad \textit{and} \quad \mathbb{E}\,|X_{\mathbf{n}}|^q < \infty, \qquad \mathbf{n} \in \mathbb{N}^d.$$

Then for $q \ge 2$, there exists a positive constant $C = C(q)$ such that

$$\mathbb{E} \max_{\mathbf{k} \preceq \mathbf{n}} |S_{\mathbf{k}}|^q \le C \Big\{ \Big(\log_2 |\mathbf{n}|\Big)^{qd} \Big(\sum_{\mathbf{k} \preceq \mathbf{n}} \mathbb{E}\, X_{\mathbf{k}}^2\Big)^{q/2} + \sum_{\mathbf{k} \preceq \mathbf{n}} \mathbb{E}\,|X_{\mathbf{k}}|^q \Big\}, \qquad \mathbf{n} \in \mathbb{N}^d.$$

As we will see in a moment, this lemma allows to supplement the theorem proven in the work of Peligrad and Gut [104], dedicated to the case of random fields with the same distribution and structure of ρ^*-mixing. We weaken the assumption of a identical distributions and examine random fields with the structure of negative association. The following theorem speaks about this.

Theorem 2.1.3. ([84], Theorem 2.1) *Let $\{X_n, n \in \mathbb{N}^d\}$ be a negatively associated random field. Let $\alpha p > 1$, $\alpha > \frac{1}{2}$ and for some $q \geq 2$*

(i) $\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha p-2} \sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{P}[|X_{\mathbf{k}}| > |\mathbf{n}|^{\alpha}] < \infty,$

(ii) $\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\left(|X_{\mathbf{k}}|^q \mathbb{I}[|X_{\mathbf{k}}| \leq |\mathbf{n}|^{\alpha}]\right) < \infty,$

(iii) $\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha(p-q)-2} (\log_2 |\mathbf{n}|)^{qr} \left(\sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\left(X_{\mathbf{k}}^2 \mathbb{I}[|X_k| \leq |\mathbf{n}|^{\alpha}]\right)\right)^{q/2} < \infty,$

(iv) $\max_{\mathbf{k}\preceq\mathbf{n}} \left|\sum_{i\preceq\mathbf{k}} \mathbb{E}\left(X_i \mathbb{I}[|X_i| \leq |\mathbf{n}|^{\alpha}]\right)\right| = o(|\mathbf{n}|^{\alpha}).$

Then we have

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha p-2} \mathbb{P}\left(\max_{\mathbf{k}\preceq\mathbf{n}} |S_k| > \varepsilon |\mathbf{n}|^{\alpha}\right) < \infty \text{ for all } \varepsilon > 0. \tag{7}$$

The results of this type are known in the literature as Baum-Katz theorems, sometimes also referred to as the Hsu-Robbins-Erdős-Spitzer-Baum-Katz theorem. The power with the $\alpha p - 2$ exponent visible in the series appearing in the assertion, in relation to the information about its convergence, allows to estimate the order of magnitude of probabilities $\mathbb{P}(\max_{j\preceq\mathbf{n}} |S_j| > \varepsilon|\mathbf{n}|^{\alpha})$ for "large" $|\mathbf{n}|$.

The assumptions of the previous theorem are quite complicated. To illustrate them, in the work [84], following the notion of weak mean dominance, known in the literature, we introduced a class of weak mean bounded fields. As it turns out, in this class a much simpler necessary and sufficient condition for the convergence, such as (7), can be deduced from the Theorem 2.1.3. Let us start with the definition mentioned above.

Definition 2.1.4. ([84], Def. 1.4) *Random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is weakly mean bounded (WMB) by random variable ξ (possibly defined on different probability space) if there exist some constants $\kappa_1, \kappa_2 > 0$, such that for all $x > 0$ and $\mathbf{n} \in \mathbb{N}^d$*

$$\kappa_2 \mathbb{P}(|\xi| > x) \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{P}(|X_{\mathbf{k}}| > x) \leq \kappa_1 \mathbb{P}(|\xi| > x).$$

If only the right hand side of the inequality is fulfilled we say that the random field $\{X_n, n \in \mathbb{N}^d\}$ satisfies weak mean dominated condition (WMD condition). Let us note that every random field composed of variables with identical distributions is weakly mean bounded (in this case inequalities by definition turn into equations with constants $\kappa_1 = \kappa_2 = 1$).

The class of WMB random fields also includes a much larger class, random variables meeting the condition of regular cover, introduced by Pruss in [106] or its less restrictive version, for example used in [73], Theorem 2.1. Since independent variables are negatively associated, the following theorem is an important generalization of the result of Gut (see [49], Theorem 3.1), which dealt with the case of independent random variables with the same distribution.

Theorem 2.1.5. ([84], Theorem 2.2) *Let α and p be constants such that $\alpha p > 1$ and $\alpha > \frac{1}{2}$, $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a negatively associated random field, weakly bounded by random variable ξ. If $p \geq 1$, we additionally assume that, $\mathbb{E}\, X_{\mathbf{n}} = 0$, for all $\mathbf{n} \in \mathbb{N}^d$. Then the following condition are equivalent:*

(A) $\mathbb{E}\, |\xi|^p \big(\log^+ |\xi|\big)^{d-1} < \infty$;

(B) $\sum\limits_{\mathbf{n} \in \mathbb{N}^d} |\mathbf{n}|^{\alpha p - 2} \mathbb{P}\left(\max_{k \preceq \mathbf{n}} |S_k| > \varepsilon |\mathbf{n}|^{\alpha}\right) < \infty$.

Let us add that in [84], for ρ^*-mixing random fields, we get the results in the form of the above-formulated statements 2.1.3 and 2.1.5, generalizing Theorems 2 and 5 from the work of Peligrad and Gut [103] in the case of ρ^*-mixing random fields with nonidentical distributions.

2.2. Results for martingale random fields

In this subsection, we would like to show that a (7) type assertion can also be obtained for martingale random fields. We will start with the lemma, which is a kind of Burkholder inequality. It is one of the few maximal inequalities that have their equivalents for the martingale random fields. The other inequalities are those of Doob (for moments of $p > 1$) and Hajek-Renyi. Due to the fact that its formulation contains the concept of the field of increments, let us recall (see [84], p. 580) that for any field $\{Z_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ there is a field $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$, determined almost surely, such that

$$Z_{\mathbf{n}} = \sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}, \qquad \mathbf{n} \in \mathbb{N}^d.$$

Such fields are an important tool for studying fields with a martingale structure and are known as fields of differences (martingale differences).

Lemma 2.2.1. ([84], Lemma 4.1)

be the corresponding field of martingale differences, $q-$integrable. Then there exist finite and positive constant C depending only on q and d such that

$$E\Big(\max_{\mathbf{k} \preceq \mathbf{n}} |Z_{\mathbf{n}}|^q\Big) \leq CE\Big(\sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}^2\Big)^{q/2}.$$

This lemma allows to prove the Baum-Katz type theorem for martingale random fields. Before we formulate this result, we must introduce new notations, which we will also use in the next sections. Set:

- $D := \{1, 2, ..., d\}$, $\emptyset \neq J \subseteq D$ i $CJ := D \setminus J$;
- for all $(n_1, ..., n_d) \in \mathbb{N}^d$, $\mathfrak{F}_{\mathbf{n}}^J := \bigvee_{(n_j \in \mathbb{N}, j \in CJ)} \mathfrak{F}_{\mathbf{n}}$ and if $J = \{j\}$ then $\mathfrak{F}_{\mathbf{n}}^J$ is denoted by $\mathfrak{F}_{\mathbf{n}}^j$;
- $\mathcal{G}_{\mathbf{n}} := \bigvee_{j=1}^{d} \mathfrak{F}_{\mathbf{n}}^j$;
- $\widetilde{\mathfrak{F}}_{\mathbf{n-1}} := \mathcal{G}_{\mathbf{n-1}} \wedge \mathfrak{F}_{\mathbf{n}}$, where $\mathbf{n} - \mathbf{1} = (n_1 - 1, n_2 - 1, ..., n_d - 1)$.

Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be the field of martingale differences of martingale $\{(S_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}),\ \mathbf{n} \in \mathbb{N}^d\}$ whose filtration fulfills the conditional independence property (F4). We are now ready to formulate the previously announced result.

Theorem 2.2.2. ([84], Theorem 4.1) *Let α, p and q be a constants such that $\alpha p > 1$, $p > 1$ and $\alpha > \frac{1}{2}$. Assume condition (i) of Theorem 2.1.3 and depending on q, one of the following condition:*

a) $$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha(p-q)-3+q/2} \sum_{k\preceq\mathbf{n}} \mathbb{E}\left(|X_k|^q \mathbb{I}[|X_k| \le |\mathbf{n}|^\alpha]\right) < \infty, \ \ dla \ \ q \ge 2;$$

b) $$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{k\preceq\mathbf{n}} \mathbb{E}\left(|X_k|^q \mathbb{I}[|X_k| \le |\mathbf{n}|^\alpha]\right) < \infty, \ \ dla\ 1 < q < 2.$$

Furthermore, if for all $\varepsilon > 0$

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha p-2} \mathbb{P}\Big[\max_{k\preceq\mathbf{n}} \Big|\sum_{i\preceq k} \mathbb{E}\left(X_i \mathbb{I}[|X_i| \le |\mathbf{n}|^\alpha] \big| \widetilde{\mathfrak{F}}_{\mathbf{n-1}}\right)\Big| > \varepsilon |\mathbf{n}|^\alpha\Big] < \infty, \qquad (8)$$

then (7) is satisfied.

The aforementioned theorem is the equivalent to the result for random fields with the structure of a negative association (cf. Theorem 2.1.3) and the structure of ρ^*-mixing (see [84], Theorem 3.1). In the class of independent random fields, common to the three dependence structures mentioned above, condition (8) is reduced to condition (iv) in Theorem 8 and Theorem 3.1 from [84]. We give such a version of the result in the following conclusion

Corollary 2.2.3. ([84], Corollary 4.1) *Let $\{X_\mathbf{n},\ \mathbf{n} \in \mathbb{N}^d\}$ be a field of independent random variables with zero-mean value, α and p be a constants such that $p \ge 1$, $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Furthermore assume conditions (i) of Theorem 2.1.3 and either condition (a) or (b) of Theorem 2.2.2, with some constant $q > 1$. Then, if*

$$\frac{1}{|\mathbf{n}|^\alpha} \max_{j\preceq\mathbf{n}} \sum_{i\preceq j} \mathbb{E}\left(X_i \mathbb{I}[|X_i| \le |\mathbf{n}|^\alpha]\right) \longrightarrow 0, \ \ as \ \ \max \mathbf{n} \to \infty, \qquad (9)$$

is fulfilled then (7) holds.

In addition to the comment on the last two results, we note that almost sure convergence of the series

$$\sum_{k\preceq n} \frac{\mathbb{E}\left(|\ X_k\ |^p\ \big|\widetilde{\mathfrak{F}}_{\mathbf{n-1}}\right)}{|\mathbf{k}|^{\alpha p}}$$

is sufficient for condition (8) and the convergence of the above series in the form with an unconditional expected values sufficient for condition (9); moreover, if the field of independent random variables with zero-mean values meets WMD, then condition (A) appearing in Theorem 2.1.5 implies all the assumptions of Corollary 2.2.3, which imply assertion (7). This is the content of Corollary 4.2 given in [84].

Chapter 3

Fuk-Nagaev, Kahane-Hoffmann-Jørgensen inequalities and Baum-Katz type theorems

Numerous results generalizing the Baum-Katz theorem (B-K) are proved using the methods of symmetrization/desymmetrization and different versions of the Kahane-Hoffmann-Jørgensen inequality (K-H-J); works [53], [58] or [72] are examples in which such a command scheme was used. The K-H-J inequality has no equivalent for sequences of dependent random variables, much less for random fields with dependence structure. The difficulties we encounter in attempts to prove it indicate rather that for dependent random variables it may be untrue. The weaker version of Kahane-Hoffmann-Jørgensen inequalities proposed by us, for random fields with a dependence structure, is still useful for the study of complete and almost sure convergence. The key outcome that allows this idea to be realized is the Fuk-Nagaev inequality (F-N); we are proving them for random fields with a negative dependence and martingale structure. a direct inspiration for these studies was the desire to extend the results of Gut and Stadtmüller, included in [53], into random fields with different structures of dependence and non identically distribution.

3.1. Results for martingale random fields

In this subsection we will present the Baum-Katz theorems from which we get information about the rate of convergence for the so-called asymmetric strong laws of large numbers (cf. [54]); that differs from the results of the second chapter, in that now the normalizing field has now the form $|\mathbf{n}^{\boldsymbol{\alpha}}|$, where $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, ..., \alpha_d) \in (\frac{1}{2}, \infty)^d$ and $\mathbf{n}^{\boldsymbol{\alpha}} := (n_1^{\alpha_1}, n_2^{\alpha_2}, ..., n_d^{\alpha_d})$. To receive such results – Burkholder's or Rosenthal's inequalities, which we successfully used in the second section, are not enough; here we must use a much more perfect tool – the Kahane-Hoffmann-Jørgensen inequalities. To formulate the results announced, we need additional concepts and notions related to martingale random fields. We will use the same references of notions that are used in the literature (cf., [12] and [62]):

(X2) $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ is adapted to filtration $\{\mathfrak{F}_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$;

(X3) $\mathbb{E}\,(X_{\mathbf{n}}|\widetilde{\mathcal{F}}_{\mathbf{n-1}}) \leq 0$ a.s., for all $\mathbf{n} \in N^d$;

(X3') $\mathbb{E}\,(X_{\mathbf{n}}|\widetilde{\mathcal{F}}_{\mathbf{n-1}}) = 0$ a.s., for all $\mathbf{n} \in N^d$;

(F2) filtration is complete in $(\Omega, \mathfrak{F}, P)$;

(F5) $\mathbb{E}\,[\mathbb{E}\,(Y|\mathfrak{F}_{\mathbf{n}})|\mathcal{G}_{\mathbf{n-1}}] = \mathbb{E}\,(Y|\widetilde{\mathfrak{F}}_{\mathbf{n-1}})$ a.s., for all $\mathbf{n} \in N^d$ and arbitrary bounded random variable Y.

The Fuk-Nagaev inequality for martingale and reversed martingale random fields was proved in the papers [78] and [80], for the case d = 2 and expanded in [12] for the case of $d > 2$. The authors of these works formulated assumptions based on at least a second conditional moment. The results presented in [85] were obtained assuming conditional moments of order r, $1 \leq r \leq 2$.

Let

$$\{b_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\}, \quad \{d_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\}, \quad \{\lambda_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\} \text{ and } \{m_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\}$$

be such families of positive numbers that for all $\mathbf{k} \in \mathbb{N}^d$ the following inequalities are almost surely satisfied:

$$\begin{aligned} &\mathbb{E}\left(|X_{\mathbf{k}}|^r \mathbb{I}[|X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) \leq b_{\mathbf{k}}^r, && \mathbb{E}\left(X_{\mathbf{k}} \mathbb{I}[X_{\mathbf{k}} > -y] | \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) \leq d_{\mathbf{k}}, \\ &\mathbb{E}\left(|X_{\mathbf{k}}|^r | \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) \leq m_{\mathbf{k}}^r, && \mathbb{E}\left(|X_{\mathbf{k}}| \mathbb{I}[|X_{\mathbf{k}}| > y] | \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) \leq \lambda_{\mathbf{k}}. \end{aligned} \tag{10}$$

Let us denote:

$$\begin{aligned} B_{\mathbf{n}}^r &:= \sum_{\mathbf{k} \preceq \mathbf{n}} b_{\mathbf{k}}^r, & D_{\mathbf{n}} &:= \sum_{\mathbf{k} \preceq \mathbf{n}} d_{\mathbf{k}}, \\ M_{\mathbf{n}}^r &:= \sum_{\mathbf{k} \preceq \mathbf{n}} m_{\mathbf{k}}^r, & \Lambda_{\mathbf{n}} &:= \sum_{\mathbf{k} \preceq \mathbf{n}} \lambda_{\mathbf{k}}. \end{aligned} \tag{11}$$

The notations and conditions introduced allow to formulate a theorem, the assertion of which is the Fuk-Nagaev inequality for martingale random fields.

Theorem 3.1.1. ([85], Theorem 4.1) *Suppose, that $\{\mathfrak{F}_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ satisfies condition (F1),(F2), (F4) and in the case $d > 2$, condition (F5), whereas random field $\{X_{\mathbf{n}}, \in \mathbb{N}^d\}$ fulfils (X2), (X3) and (10). Furthermore, if the constants x, y and r are such that $x, y > 0$ and $1 \leq r \leq 2$, then*

$$\begin{aligned} \mathbb{P}(\max_{\mathbf{k} \preceq \mathbf{n}} S_{\mathbf{k}} \geq x) \leq{}& \mathbb{P}(\max_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \geq y) \\ &+ e^{d-1} \exp\left\{ \frac{x}{y} - \left(\frac{x - D_{\mathbf{n}}}{y} + \frac{B_{\mathbf{n}}^r}{y^r} \right) \ln\left[\frac{x y^{r-1}}{B_{\mathbf{n}}^r} + 1 \right] \right\}. \end{aligned} \tag{12}$$

If we assume (X3') instead of (X3), we have

$$\begin{aligned} \mathbb{P}(\max_{\mathbf{k} \preceq \mathbf{n}} |S_{\mathbf{k}}| \geq x) \leq{}& \mathbb{P}(\max_{\mathbf{k} \preceq \mathbf{n}} |X_{\mathbf{k}}| \geq y) \\ &+ 2e^{d-1} \exp\left\{ \frac{x}{y} - \left(\frac{x - D_{\mathbf{n}}}{y} + \frac{B_{\mathbf{n}}^r}{y^r} \right) \ln\left[\frac{x y^{r-1}}{B_{\mathbf{n}}^r} + 1 \right] \right\}. \end{aligned} \tag{13}$$

Proof. (sketch) Let us put

$$\widetilde{X_{\mathbf{k}}} = X_{\mathbf{k}} I[X_{\mathbf{k}} \leq y], \ \ \widetilde{S_{\mathbf{k}}} = \sum_{\mathbf{k}\leq\mathbf{n}} \widetilde{X_{\mathbf{k}}} \text{ and}$$
$$Z_{\mathbf{k}} = \widetilde{X_{\mathbf{k}}} - E(\widetilde{X_{\mathbf{k}}} \mid \widetilde{\mathfrak{F}}_{\mathbf{n-1}}), \ \ T_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} Z_{\mathbf{k}}.$$

Obviously, we have

$$P(\max_{\mathbf{k}\leq\mathbf{n}} S_{\mathbf{k}} \geq x) \leq P(\max_{\mathbf{k}\leq\mathbf{n}} \widetilde{S_{\mathbf{k}}} \geq x) + P(\max_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}} \geq y). \tag{14}$$

From (X3) implies, that $Z_{\mathbf{k}} \geq \widetilde{X_{\mathbf{k}}}$ a.s. and since $\alpha > 1$, $h > 0$

$$P(\max_{\mathbf{k}\leq\mathbf{n}} \widetilde{S_{\mathbf{k}}} \geq x) \leq P(\max_{\mathbf{k}\leq\mathbf{n}} e^{\alpha h T_{\mathbf{k}}} \geq e^{\alpha h x}). \tag{15}$$

Let us observe, that $\{(e^{\alpha h T_{\mathbf{k}}}, \widetilde{\mathfrak{F}}_{\mathbf{k}}), \mathbf{k} \leq \mathbf{n}\}$ is positive submartingale. Denote $\mathbf{k}(j) = (k_1, k_2, \cdots, k_{j-1}, k_{j+1}, \cdots, k_d)$ for $\mathbf{k} \in \mathbb{N}^d$ and $1 \leq j \leq d$, thus $\{\max_{\mathbf{k}(d)\leq\mathbf{n}(d)} e^{\alpha h T_{\mathbf{k}}}, 1 \leq k_d \leq n_d\}$ is positive d-sumbartingale with respect to $\{\mathfrak{F}^d_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$. By application of standard Doob inequality to d-submartingale and Doob inequality for submartingale random field, cf.Shorack et al. [110]

$$\begin{aligned} P(\max_{\mathbf{k}\preceq\mathbf{n}} e^{\alpha h T_{\mathbf{k}}} \geq e^{\alpha h x}) &\leq e^{-\alpha h x} E\left(\max_{\mathbf{k}(d)\preceq\mathbf{n}(d)} (e^{h T_{\mathbf{k}(d) n_d}})^{\alpha}\right) \\ &\leq \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha(d-1)} e^{-\alpha h x} E e^{\alpha h T_{\mathbf{n}}}. \end{aligned} \tag{16}$$

Furthermore, we need estimations:

- $E(e^{\alpha h \widetilde{X_{\mathbf{k}}}} \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}) = E(e^{\alpha h \widetilde{X_{\mathbf{k}}}} I[\widetilde{X_{\mathbf{k}}} < -y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}) +$
  $E(e^{\alpha h \widetilde{X_{\mathbf{k}}}} I[\mid \widetilde{X_{\mathbf{k}}} \mid \leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}) = I_{10} + I_{11}$,
- $I_{10} \leq E(I[X_{\mathbf{k}} < -y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})$,

- $I_{11} \leq \frac{e^{\alpha hy-1-\alpha hy}}{y^2} E\left(X_{\mathbf{k}}^2 I[0<\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) +$
  $\alpha h E(X_{\mathbf{k}} I[\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}) + E(I[\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})$
  $\leq \frac{e^{\alpha hy-1-\alpha hy}}{y^r} E\left(\mid X_{\mathbf{k}} \mid^r I[0<\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) +$
  $\alpha h E(X_{\mathbf{k}} I[\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})$.

Thus

$$\begin{aligned}
&E(e^{\alpha h Z_{\mathbf{k}}} \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}) \leq \\
&e^{-\alpha h E(X_{\mathbf{k}} I[X_{\mathbf{k}}\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})} \exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} E\left(\mid X_{\mathbf{k}} \mid^r I[0<\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) +\right.\\
&\left.\alpha h E(X_{\mathbf{k}} I[\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})\right\} \leq \\
&\exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} E\left(\mid X_{\mathbf{k}} \mid^r I[0<\mid X_{\mathbf{k}} \mid\leq y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}}\right) -\right.\\
&\left.\alpha h E(X_{\mathbf{k}} I[X_{\mathbf{k}} < -y] \mid \widetilde{\mathfrak{F}}_{\mathbf{k-1}})\right\} \leq \exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} b_{\mathbf{k}}^r + \alpha h d_{\mathbf{k}}\right\}.
\end{aligned} \tag{17}$$

Now, furnishing $\{\mathbf{k} : \mathbf{k} \preceq \mathbf{n}\}$ with a total order and using property (F5), we have

$$E e^{\alpha h T_{\mathbf{n}}} \preceq \exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} \sum_{\mathbf{k}\preceq\mathbf{n}} b_{\mathbf{k}}^r + \alpha h \sum_{\mathbf{k}\preceq\mathbf{n}} d_{\mathbf{k}}\right\}. \tag{18}$$

Combining (14), (15), (16) and (18) we get

$$\begin{aligned}
&P(\max_{\mathbf{k}\preceq\mathbf{n}} S_{\mathbf{k}} \geq x) \leq \\
&P(\max_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} \geq y) + \left(\frac{\alpha}{\alpha-1}\right)^{\alpha(d-1)} e^{-\alpha hx} \exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} B_{\mathbf{k}}^r + \alpha h D_{\mathbf{k}}\right\} \leq \\
&P(\max_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} \geq y) + e^{d-1} \exp\left\{\frac{e^{\alpha hy-1-\alpha hy}}{y^r} B_{\mathbf{k}}^r + \alpha h D_{\mathbf{k}} - \alpha hx\right\}.
\end{aligned} \tag{19}$$

Setting,

$$\alpha h = \frac{1}{y} \ln[\frac{xy^{r-1}}{B_{\mathbf{n}}^r 1} + 1]$$

one can obtain (12).

To prove (13), we set:

$$Y_{\mathbf{k}} = -X_{\mathbf{k}} \text{ and } U_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n}} Y_{\mathbf{k}}.$$

Obviously, $\{(U_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$ is martingale random field satisfying assumption of our theorem. Furthermore, denote

$$\widetilde{Y}_{\mathbf{k}} = Y_{\mathbf{k}} I[Y_{\mathbf{k}} \leq y] \text{ and } \widetilde{U}_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n}} \widetilde{Y}_{\mathbf{k}}.$$

Then, by standard estimation we have

$$P(\max_{\mathbf{k}\preceq\mathbf{n}} \mid S_{\mathbf{k}} \mid\geq x) \leq P(\max_{\mathbf{k}\leq\mathbf{n}} \mid X_{\mathbf{k}} \mid\geq y) + P(\max_{\mathbf{k}\preceq\mathbf{n}} \widetilde{S_{\mathbf{k}}} \geq x) + P(\max_{\mathbf{k}\preceq\mathbf{n}} \widetilde{U_{\mathbf{k}}} \geq x),$$

then similarly, as in the first part of the proof we obtain (13).

□

Inequalities (12) and (13), called Fuk-Nagaev inequalities, allow to obtain a tool for proving Baum-Katz theorems – the Kahane-Hoffmann-Jørgensen inequality.

Lemma 3.1.2. ([85], Lemma 4.2) *Let $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$ satisfies assumption of Theorem 3.1.1 with condition (X3'), then there exist the non-negative constants C depending only on d and j such that for all $x > 0$ and $j > 0$*

$$\mathbb{P}(\max_{\mathbf{k}\preceq\mathbf{n}} |S_{\mathbf{k}}| \geq x) \leq \mathbb{P}\left(\max_{\mathbf{k}\preceq\mathbf{n}} |X_{\mathbf{k}}| \geq \frac{x}{j}\right) + C\left(\frac{1}{x^r} M_{\mathbf{n}}^r\right)^j \tag{20}$$

holds.

Application of Lemma 3.1.2 affords two theorems. The first of these, the Baum-Katz theorem, defines the rate of convergence in Marcinkiewicz strong law of large numbers, with asymmetric normalization. It is a martingale version of the result of Gut and Stadtmüller obtained for independent random variables with the same distribution (see [53], Theorem 1.3); it is therefore a significant generalization of this result.

Theorem 3.1.3. ([85], Theorem 4.3) *Let $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}),\ \mathbf{n} \in \mathbb{N}^d\}$ satisfies assumptions of Theorem 3.1.1 and WMD condition with random variable ξ, moreover assume that for some constants α_1 and r satisfying the following inequalities $\alpha_1 r > 1$ and $\alpha_1 > 1/2$, there exist $\mathbf{n_0} \in \mathbb{N}^d$ and the constant M depends only on r such that $\frac{1}{|\mathbf{n}|} M^r_{\mathbf{n}} \leq M$, for all $\mathbf{n} \not\preceq \mathbf{n_0}$. Then, if*

$$\mathbb{E}\,|\xi|^r (\log^+ |\xi|)|^{p-1} < \infty, \tag{21}$$

then

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha_1 r-2} \mathbb{P}(\max_{\mathbf{k}\preceq\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty, \quad \textit{for all} \quad \varepsilon > 0, \tag{22}$$

where α_1 is the smallest coordinate of vector $\boldsymbol{\alpha}$.

The second theorem that we can get using Lemma 3.1.2 is a generalizations of the results of Ghosal and Chandra (cf. [47], Theorem 1 (b) and 2) to martingale random fields, with less restrictive moment requirements and more general assumptions regarding the matrix of random variables – even in a one-dimensional case. Here is the content of this statement.

Theorem 3.1.4. ([85], Theorem 4.4) *Let $\mathbf{k_n}$ be a field of such elements of $\mathbb{N}^d$, that $|\mathbf{k_n}|$ tends to infinity as $\max \mathbf{n} \to \infty$; $\{X_{\mathbf{n},\mathbf{i}}, \mathbf{i} \preceq \mathbf{k_n},\ \mathbf{n} \in \mathbb{N}^d\}$ be a d-dimensional array of rowwise martingale differences with respect to family of σ-algebra $\{\mathfrak{F}_{\mathbf{n},\mathbf{i}}, \mathbf{i} \preceq \mathbf{k_n},\ \mathbf{n} \in \mathbb{N}^d\}$, satisfying assumptions of Theorem 3.1.1 with condition (X3'). Moreover, there exist a nonnegative real fields $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ i $\{\widetilde{M^r_{\mathbf{k_n}}}, \mathbf{n} \in \mathbb{N}^d\}\}$ such that*

(i) $\sum_{\mathbf{j}\preceq\mathbf{k_n}} \mathbb{E}\,(|X_{\mathbf{n},\mathbf{j}}|^r | \widetilde{\mathfrak{F}}_{\mathbf{n},\mathbf{j-1}}) \leq \widetilde{M^r_{\mathbf{k_n}}}$ *a.s.*,

(ii) $\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}} \mathbb{P}(\max_{\mathbf{i}\preceq\mathbf{k_n}} |X_{\mathbf{n},\mathbf{i}}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$,

(iii) *there exists the constant $j > 0$ such that* $\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}} \left(\widetilde{M^r_{\mathbf{k_n}}}\right)^j < \infty$,

then

$$\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}} \mathbb{P}(\max_{\mathbf{l}\preceq\mathbf{k_n}} |\sum_{\mathbf{i}\preceq\mathbf{l}} X_{\mathbf{n},\mathbf{i}}| > \varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Examples of uses of this type of results as the Theorem 3.1.4 we can find, for example, in paper [47].

3.2. Results for negatively dependent random fields

In this subsection we show that Baum-Katz theorems with asymmetric normalization can also be obtained for random fields with the structure of negative dependence. Before we proceed to the formulation of the most important results, we will recall the definitions of random fields with such a structure.

Definition 3.2.1. *A finite family of random variables* $\{X_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n},\ \mathbf{n} \in \mathbb{N}^d\}$ *is said to be negatively dependent (ND) if*

$$\mathbb{P}\left[\bigcap_{\mathbf{k}\preceq\mathbf{n}} (X_{\mathbf{k}} \le x_{\mathbf{k}})\right] \le \prod_{\mathbf{k}\preceq\mathbf{n}} \mathbb{P}(X_{\mathbf{k}} \le x_{\mathbf{k}})$$

and

$$\mathbb{P}\left[\bigcap_{\mathbf{k}\preceq\mathbf{n}} (X_{\mathbf{k}} > x_{\mathbf{k}})\right] \le \prod_{\mathbf{k}\preceq\mathbf{n}} \mathbb{P}(X_{\mathbf{k}} > x_{\mathbf{k}}),$$

for all $x_{\mathbf{k}} \in \mathbb{R}$, $\mathbf{k} \preceq \mathbf{n}$.

An infinite family is ND if every finite subfamily is ND.

The condition of the negative dependence of a random field is obviously less restrictive than the condition of negative association. It can be shown (see [61]) that the class of negatively associated random fields is included in the class of negatively dependent random fields.

Presentation of the results will start with several lemmas which are crucial in the proof of the condition sufficient in Theorem 3.2.12, but also very well explains the necessity of additional assumptions in Baum-Katz theorems for random fields with different distributions.

Lemma 3.2.2. ([85], Lemma 2.1) *If* $\prod_{\mathbf{k}\preceq\mathbf{n}}(1-a_{\mathbf{k},\mathbf{n}})\to 1$ *as* $|\mathbf{n}|\to\infty$, *then for a given* $0<\delta<1$ *and sufficiently large* $\mathbf{n_0}\in\mathbb{N}^d$

$$\bigwedge_{\mathbf{n}\not\preceq\mathbf{n_0}} \quad 1-\prod_{\mathbf{k}\preceq\mathbf{n}}(1-a_{\mathbf{k},\mathbf{n}})\geq(1-\delta)\sum_{\mathbf{k}\preceq\mathbf{n}}a_{\mathbf{k},\mathbf{n}}$$

For some $a>0$, let us put

$$X'_{\mathbf{i}}=X_{\mathbf{i}}I[|X_{\mathbf{i}}|\leq a],\quad X''_{\mathbf{i}}=X_{\mathbf{i}}I[|X_{\mathbf{i}}|>a],$$

and

$$\xi'=\xi I[|\xi|\leq a],\quad \xi''=\xi I[|\xi\,|>a].$$

Lemma 3.2.3. ([85], Lemma 2.2) *Let* $\{X_{\mathbf{n}},\mathbf{n}\in\mathbb{N}^d\}$ *be a field of random variables satisfying WMB condition with random variable* ξ *and constants* κ_1,κ_2. *Let* $s>0$.

(a) If $\mathbb{E}|\xi|^s<\infty$, *then* $\kappa_2\mathbb{E}|\xi|^s\leq\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\preceq\mathbf{n}}\mathbb{E}|X_k|^s\leq\kappa_1\mathbb{E}|\xi|^s$.

(b) $\kappa_2\mathbb{E}|\xi'|^s\leq\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\preceq\mathbf{n}}\mathbb{E}|X'_k|^s\leq\kappa_1\mathbb{E}|\xi'|^s$.

(c) $\kappa_2\mathbb{E}|\xi''|^s\leq\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\preceq\mathbf{n}}\mathbb{E}|X''_k|^s\leq\kappa_1\mathbb{E}|\xi''|^s$.

The following properties of ND random variables, proved by Bozorgnia et al. [13], for sequences of r.v., obviously hold true for ND random fields.

Lemma 3.2.4. ([85], Lemma 2.3) *Let* $\{X_k,\mathbf{k}\preceq\mathbf{n}\}$ *be a field of ND random variables and* $\{f_{\mathbf{k}},\mathbf{k}\ \mathbf{n}\}$ *a family of Borel functions, which all are non-decreasing (non-increasing), then*
(a) $\{f(X_{\mathbf{k}}),\mathbf{k}\preceq\mathbf{n}\}$ *is a ND random field,*
(b) if additionally, $X_{\mathbf{k}}$ *are non-negative, we have*

$$\mathbb{E}\left(\prod_{\mathbf{k}\preceq\mathbf{n}}X_{\mathbf{k}}\right)\leq\prod_{\mathbf{k}\preceq\mathbf{n}}\mathbb{E}X_{\mathbf{k}}.$$

Lemma 3.2.5. ([85], Lemma 2.4) *Assume, that* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *is a field of zero mean, square integrable ND random variables WMD by random variable* ξ *and such that* $\mathbb{E}\xi^2 = \sigma^2 < \infty$*, then*

(a) $$\mathbb{E}(\sum_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}})^2 \leq \kappa_1\sigma^2|\mathbf{n}|,$$

if additionally $\mathbb{P}(X_{\mathbf{k}} \leq b) = 1$ *for every* $\mathbf{k} \preceq \mathbf{n}$, *then*

(b) $$\mathbb{P}(\sum_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} > x) \leq e^{-tx+\kappa_1\sigma^2|\mathbf{n}|}$$

for all $x, b > 0$ *and* $0 < t < \frac{1}{b}$.

The next three lemmas, we present, are used to prove Lemma 3.2.9 but also they can be very useful tools in the area of limit theorems for random fields. These results are the version of the theorem for subsequences in a metric space for sequences with indices belong to partially ordered sets.

Lemma 3.2.6. *If there exists an element* $a \in Y$*, such that from every subsequence of family* $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *of elements of metric space* (Y, d) *we can choose a subsequence that converges to* a*, then*

$$\lim_{\max\mathbf{n}\to\infty} a_{\mathbf{n}} = a.$$

Lemma 3.2.7. *Let* $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a family of elements of a metric space* (Y, d)*, thus the following conditions are equivalent:*

(i) $\lim_{\max\mathbf{n}\to\infty} a_{\mathbf{n}} = a,$

(ii) $\bigwedge_{\{\mathbf{n}_l\}\subseteq\mathbb{N}^d} \left(\{\mathbf{n}_l\} \text{ increasing} \Rightarrow \lim_{l\to\infty} a_{\mathbf{n}_l} = a\right)$

Remark 1. Lemma 3.2.7 means that we can straightforward transfer metrizable convergence results for random variables to convergence of random fields as $\max\mathbf{n} \to \infty$. This tool is also very helpful to study the convergence of partial sums, it is demonstrated in the proof of Lemma 3.2.9 (see proof of Lemma 2.4 in [83]). Obviously, convergence in the sense $\max\mathbf{n} \to \infty$ implies convergence in the sense

$\min \mathbf{n} \to \infty$, thus we have another sufficient condition without completeness of Y as it was assumed in Lemma V-1-1 of [102].

Lemma 3.2.8. *Let* $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a family of non-negative real numbers and* $S_{\mathbf{n}} = \sum_{\mathbf{k} \preceq \mathbf{n}} a_{\mathbf{k}}$*, then the following conditions are equivalent:*

(i) $\lim_{\min \mathbf{n} \to \infty} S_{\mathbf{n}} = S$,

(ii) $\bigwedge_{\{\mathbf{n}_k\} \subseteq \mathbb{N}^d} \left(\{\mathbf{n}_k\} \text{ increasing} \quad \Rightarrow \lim_{k,l \to \infty} \mid S_{\mathbf{n}lb_k} - S_{\mathbf{n}_l} \mid = 0 \right).$

Lemma 3.2.9. ([83], Lemma 2.4) *Let* $\{a_{\mathbf{k},\mathbf{n}},\ \mathbf{k}, \mathbf{n} \in \mathbb{N}^d\}$ *be a* d*-dimensional array of real numbers, non-decreasing with respect to* $|\mathbf{n}|$ *and such that*
$0 < a_{\mathbf{k},\mathbf{n}} \leq 1$, $1 - a_{\mathbf{n},\mathbf{n}} = o(\frac{1}{|\mathbf{n}|})$, *then*

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{1}{|\mathbf{n}|} \left(1 - \prod_{\mathbf{k} \preceq \mathbf{n}} a_{\mathbf{k},\mathbf{n}} \right) < \infty \quad \textit{implies} \quad \lim_{\max \mathbf{n} \to \infty} \prod_{\mathbf{k} \preceq \mathbf{n}} a_{\mathbf{k},\mathbf{n}} = 1. \tag{23}$$

The above result is a generalization of the second part of the Lemma A4.2 from Gut's monograph [56] and allows to obtain a complete convergence results for the field $\max_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}$ of random variables with non identically distributed. Lemma 3.2.9 was proved for the purpose of the proof of Theorem 3.2.12, but it can be generalized to a form in which we consider the series

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{1}{b_{\mathbf{n}}} (1 - \prod_{\mathbf{l} \preceq \mathbf{n}} a_{\mathbf{l},\mathbf{n}}),$$

which will allow further extensions of the Baum-Katz theorem for random fields, for example towards theorem 11.1 from chapter VI of the monograph [56].

Let us now turn to the main results of this subsection; before we do it, to simplify the expression, let us put

$$\widehat{M}^r_{\mathbf{n}} := \sum_{\mathbf{k} \preceq \mathbf{n}} \mathbb{E} \, |X_{\mathbf{k}}|^r.$$

Lemma 3.2.10. ([85], Lemma 2.5) *Let $\{X_{\mathbf{n}},\, n \in \mathbb{N}^d\}$ be a field of zero mean ND random variables with finite an absolute r-th moment, $1 \leq r \leq 2$, then there exist constants $C > 0$ depends only on d and j such that for all $x > 0$ and $j > 0$*

$$\mathbb{P}(|S_{\mathbf{n}}| > x) \leq \mathbb{P}\left(\max_{\mathbf{k} \preceq \mathbf{n}} |X_{\mathbf{k}}| > \frac{x}{j}\right) + C\left(\frac{1}{x^r}\widehat{M}^r_{\mathbf{n}}\right)^j$$

holds.

The conclusion of Lemma 3.2.10 this is a Kahane-Hoffmann-Jørgensen inequality, thanks to which we can argue the Baum-Katz theorem for random fields with a negative dependence structure – we quote two such statements below, following the next Lemma.

Lemma 3.2.11. ([85], Lemma 2.6) *Let ξ be a random variable such that $\mathbb{E} \mid \xi \mid^{\frac{1}{\alpha_1}} (\log_+ \mid \xi \mid)^{p-1} < \infty$, then under our setting with $\alpha_1 > \frac{1}{2}$*

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{\mathbb{E} \mid \xi \mid^2 I[\mid \xi \mid \leq \mid \mathbf{n}^{\boldsymbol{\alpha}} \mid]}{\mid \mathbf{n}^{\boldsymbol{\alpha}} \mid^2} < \infty.$$

Theorem 3.2.12. ([83], Theorem 3.3, [85], Theorem 3.1) *Let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ be a random field of negatively dependent, zero-mean random variables, weakly mean bounded by r.v. ξ. Moreover, if some constants α_1 and r satisfy the following inequalities $r \geq 1$, $\alpha_1 > \frac{1}{2}$ i $\alpha_1 r \geq 1$ and*

$$\mathbb{E}\,|\xi|^r (\log^+ |\xi|)^{p-1} < \infty \tag{24}$$

then

$$\sum_{\mathbf{n} \in \mathbb{N}^d} |\mathbf{n}|^{\alpha_1 r - 2} \mathbb{P}(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty, \quad \textit{for all} \quad \varepsilon > 0. \tag{25}$$

Conversely,

(i) If $r > 0$, $\alpha_1 > \frac{1}{2}$, $\alpha_1 r \geq 2$ and

$$\sum_{\mathbf{n} \in \mathbb{N}^d} |\mathbf{n}|^{\alpha_1 r - 2} \mathbb{P}(\max_{\mathbf{k} \preceq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty, \; \textit{for all} \;\; \varepsilon > 0 \tag{26}$$

then (24) holds, and $\mathbb{E}\,\xi = 0$ if $r \geq 1$;

(ii) if $r>0$, $\alpha_1>\frac{1}{2}$, $\alpha_1 r\in\langle 1,2)$ *assume that*

$$\mathbb{P}(|X_{\mathbf{n}}|>|\mathbf{n}^{\boldsymbol{\alpha}}|))=o\left(\frac{1}{|\mathbf{n}|}\right) \tag{27}$$

and (26) holds for all $\varepsilon>0$ *then conclusion of (i) holds.*

Proof. ([83] and [85]) Let's start from implication (24) ⇒(25). The general idea of the proof is based on the proof of Theorem 4.1 by Gut and Stadtmüller [53], thus we sketch the proof showing differences. At the beginning, assume that $\alpha_1>\frac{1}{2}$, $\alpha_1 r>1$ and (24) holds. Applying Lemma 3.2.10 one can obtain

$$\begin{aligned}
&\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-2}P(|S_{\mathbf{n}}|>|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)\leq\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-2}\sum_{\mathbf{k}\preceq\mathbf{n}}P(|X_{\mathbf{k}}|>y)+\\
&+\frac{C}{\varepsilon^{rj}}\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-2}|\mathbf{n}^{\boldsymbol{\alpha}}|^{-jr}(\sum_{\mathbf{k}\preceq\mathbf{n}}E|X_{\mathbf{k}}|^r)^j\leq\\
&C_1\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-1}P(|\xi|>|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon^{'})+C_2\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-2+j}|\mathbf{n}^{\boldsymbol{\alpha}}|^{-jr}(E|\xi|^r)^j\\
&=I_2+I_3,\quad\text{where}\quad\varepsilon^{'}=\frac{\varepsilon}{j}.
\end{aligned} \tag{28}$$

The first sum I_2 is finite by Lemma 2.2 of [53], the second one is estimated as follows

$$\begin{aligned}
&I_2\leq C\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha_1 r-2+j}|\mathbf{n}^{\boldsymbol{\alpha}}|^{-jr}\leq\\
&C\sum_{\mathbf{n}}\prod_{i=1}^{d}n_i^{\alpha_1 r-2+(1-\alpha_1 r)j}\leq C\prod_{i=1}^{d}\sum_{n_i=1}^{\infty}n_i^{\alpha_1 r-2+(1-\alpha_1 r)j}<\infty,
\end{aligned} \tag{29}$$

since exponent in the last sum can be less than minus one, for j sufficiently large. Now, assume that $\alpha_1>\frac{1}{2}$, $\alpha_1 r=1$ and let

$$Y_{\mathbf{k},\mathbf{n}}=\min(|\mathbf{n}|^{\boldsymbol{\alpha}},|X_{\mathbf{k}}|)\operatorname{sgn}(X_{\mathbf{k}}),\ \ X_{\mathbf{k},\mathbf{n}}=X_{\mathbf{k}}I[|X_{\mathbf{k}}|\leq|\mathbf{n}|^{\boldsymbol{\alpha}}],\ \ T_{\mathbf{n}}=\sum_{\mathbf{k}\preceq\mathbf{n}}Y_{\mathbf{k},\mathbf{n}}.$$

Thus we get

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) \le$$
$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|T_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}} - T_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) = I_4 + I_5 \tag{30}$$

The first sum can be estimated by applying Chebyshev inequality, Lemma 3.2.5 and 3.2.3, WMD condition consecutively:

$$I_4 \le \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{E(T_{\mathbf{n}} - ET_{\mathbf{n}})^2}{\varepsilon^2 |\mathbf{n}^{\boldsymbol{\alpha}}|^2} \le C \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{ET_{\mathbf{n}}^2}{|\mathbf{n}^{\boldsymbol{\alpha}}|^2} \le$$
$$C(\sum_{\mathbf{n}} [\frac{1}{|\mathbf{n}|} \frac{\sum_{\mathbf{k} \preceq \mathbf{n}} EX_{\mathbf{k},\mathbf{n}}^2}{|\mathbf{n}^{\boldsymbol{\alpha}}|^2} + \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \preceq \mathbf{n}} P(|X_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}})]) \le \tag{31}$$
$$C(\sum_{\mathbf{n}} \frac{E|\xi|^2 I[|\xi| \le |\mathbf{n}^{\boldsymbol{\alpha}}|]}{|\mathbf{n}^{\boldsymbol{\alpha}}|^2} + \sum_{\mathbf{n}} P(|\xi| > \mathbf{n}^{\boldsymbol{\alpha}})) \le CE|\xi|^{\frac{1}{\alpha_1}} (\log_+ |\xi|)^{p-1}.$$

The last inequality follows from Lemma 3.2.11 and Lemma 2.2 of [53] respectively. On the other hand

$$I_5 \le \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(\sum_{\mathbf{k} \preceq \mathbf{n}} |X_{\mathbf{k}}| I[|X_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|] > \varepsilon |\mathbf{n}^{\boldsymbol{\alpha}}|) \le$$
$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(\sum_{\mathbf{k} \preceq \mathbf{n}} |X_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|) \le C \sum_{\mathbf{n}} P(|\xi| > |\mathbf{n}^{\boldsymbol{\alpha}}|) < \infty, \tag{32}$$

by WMD condition and Lemma 2.2 of [53].

Now, we prove the implication (26)⇒ (24). Firstly, let us observe, that the negative and positive part of ND random variables are still ND. Thus

$$P(\max_{\mathbf{k} \preceq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) \ge P(\max_{\mathbf{k} \preceq \mathbf{n}} |X_{\mathbf{k}}| > 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) \ge$$
$$P(\max_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}^+ > 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) = 1 - P(\bigcap_{\mathbf{k} \preceq \mathbf{n}} [X_{\mathbf{k}}^+ \le 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon]) \ge \tag{33}$$
$$1 - \prod_{\mathbf{k} \preceq \mathbf{n}} P(X_{\mathbf{k}}^+ \le 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) = 1 - \prod_{\mathbf{k} \preceq \mathbf{n}} (1 - P(X_{\mathbf{k}}^+ > 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon))$$

From (26) and (15) it's easy to see, that

$$\prod_{\mathbf{k}\preceq\mathbf{n}}(1-P(X_{\mathbf{k}}^{+}>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|))\to 1 \text{ as } |\mathbf{n}|\to\infty,$$

what is equivalent to

$$\sum_{\mathbf{k}\preceq\mathbf{n}} P(X_{\mathbf{k}}^{+}>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\to 0 \quad \text{as} \quad |\mathbf{n}|\to\infty, \tag{34}$$

confer proof of Lemma 3.2.2. Analogously, we can get

$$\sum_{\mathbf{k}\preceq\mathbf{n}} P(X_{\mathbf{k}}^{-}>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\to 0 \quad \text{as} \quad |\mathbf{n}|\to\infty. \tag{35}$$

Now, applying Lemma 3.2.2 with $a_{\mathbf{k},\mathbf{n}} = P(X_{\mathbf{k}}^{+} > \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$ and $\hat{a}_{\mathbf{k},\mathbf{n}} = P(X_{\mathbf{k}}^{-} > \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$, WMB condition and Lemma 2.2 of [53],
for sufficiently large $\mathbf{n_0}$, we have

$$\sum_{\mathbf{n}\not\preceq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2}P(\max_{\mathbf{k}\preceq\mathbf{n}}|S_{\mathbf{k}}|>\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\geq C_1\sum_{\mathbf{n}\not\preceq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2}P(\max_{\mathbf{k}\preceq\mathbf{n}}|X_{\mathbf{k}}|>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\geq$$
$$C_2\sum_{\mathbf{n}\not\preceq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2}\sum_{\mathbf{k}\preceq\mathbf{n}}P(|X_{\mathbf{k}}|>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\geq C_3\sum_{\mathbf{n}\not\preceq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-1}P(|\xi|>2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\geq$$
$$C_4E|\xi|^r(\log_+|\xi|)|^{p-1}.$$

It ends the proof of (i) thus we sketch the proof of (ii).
The negative and positive part of ND random variables are still ND, then

$$\begin{aligned}
&P(\max_{\mathbf{k}\preceq\mathbf{n}}|S_{\mathbf{k}}|>|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)\geq P(\max_{\mathbf{k}\preceq\mathbf{n}}|X_{\mathbf{k}}|>2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)\geq\\
&P(\max_{\mathbf{k}\preceq\mathbf{n}}X_{\mathbf{k}}^{+}>2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)=1-P(\bigcap_{\mathbf{k}\preceq\mathbf{n}}[X_{\mathbf{k}}^{+}\leq 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon])\geq\\
&1-\prod_{\mathbf{k}\preceq\mathbf{n}}P(X_{\mathbf{k}}^{+}\leq 2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)=1-\prod_{\mathbf{k}\preceq\mathbf{n}}(1-P(X_{\mathbf{k}}^{+}>2|\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon)).
\end{aligned} \tag{36}$$

Let $a_{\mathbf{k},\mathbf{n}} = P(X^+_{\mathbf{k}} \leq 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|))$ thus Lemma 3.2.9 implies, that

$$\prod_{\mathbf{k}\preceq\mathbf{n}}(1 - P(X^+_{\mathbf{k}} > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)) \to 1 \text{ as } \max\mathbf{n} \to \infty.$$

Analogously, we can get

$$\prod_{\mathbf{k}\preceq\mathbf{n}}(1 - P(X^-_{\mathbf{k}} > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)) \to 1 \text{ as } \max\mathbf{n} \to \infty.$$

Now, applying Lemma 3.2.2 with $a_{\mathbf{k},\mathbf{n}} = P(X^+_{\mathbf{k}} \leq \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$ and $\hat{a}_{\mathbf{k},\mathbf{n}} = P(X^-_{\mathbf{k}} \leq \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$, WMB condition and Lemma 2.2 of [53], for sufficiently large $\min\mathbf{n_0}$, we have

$$\sum_{\mathbf{n}\npreceq\mathbf{n_0}}|\mathbf{n}|^{\alpha_1 r-2}P(\max_{\mathbf{k}\preceq\mathbf{n}}|S_{\mathbf{k}}| > \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) \geq C_1\sum_{\mathbf{n}\npreceq\mathbf{n_0}}|\mathbf{n}|^{\alpha_1 r-2}P(\max_{\mathbf{k}\preceq\mathbf{n}}|X_{\mathbf{k}}| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) \geq$$
$$C_2\sum_{\mathbf{n}\npreceq\mathbf{n_0}}|\mathbf{n}|^{\alpha_1 r-2}\sum_{\mathbf{k}\preceq\mathbf{n}}P(|X_{\mathbf{k}}| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) \geq C_3\sum_{\mathbf{n}\npreceq\mathbf{n_0}}|\mathbf{n}|^{\alpha_1 r-1}P(|\xi| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) \geq$$
$$C_4 E|\xi|^r(\log_+|\xi|)|^{p-1}.$$

The second part of assertion i.e., that if $r \geq 1,\ E\xi = 0$, is rather known, confer [53] or [55]. □

The above theorem is another generalization of the result of Gut and Stadtmüller (see [53], Theorem 1.3), obtained for fields of independent random variables, having the same distribution – into random fields with nonidentical distribution and the structure of negative dependence.

At this point it is worth noting that even for sequences of independent random variables, but with different distributions, in a necessary condition, without any additional assumptions, we cannot weaken the inequality $\alpha_1 r \geq 2$. The Baum-Katz condition (26) is satisfied by random variable sequences, with subsequences behaving like a random variable with a very heavy tail, without any moment; using Lemma 3.2.10 we can construct a suitable example. An additional assumption to eliminate such subsequences is therefore necessary; it seems that one of the weakest is condition (27). On the other hand, if in Theorem 3.2.12 we assume the same distributions, of course the problem disappears and we get the following generalization of the result of Gut and Stadtmüller (cf. [53], Theorem 1.3) to the case of fields with a negative dependence structure.

Theorem 3.2.13. ([83], Theorem 3.4). *Let $\{X_n,\, n \in \mathbb{N}^d\}$ be a random field of negatively dependent, identically distributed as X random variables. If constants α_1 and r are such that: $r \geq 1$, $\alpha_1 > \frac{1}{2}$, $\alpha_1 r \geq 1$ and*

$$\mathbb{E}\,|X|^r(\log^+|X|)^{d-1} < \infty, \qquad \mathbb{E}\,X = 0, \tag{37}$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2}\mathbb{P}(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty, \quad \textit{for all} \quad \varepsilon > 0. \tag{38}$$

Conversely, if $r > 0$, $\alpha_1 > \frac{1}{2}$, $\alpha_1 r \geq 1$ and

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{\alpha_1 r-2}\mathbb{P}(\max_{\mathbf{k}\preceq\mathbf{n}}|S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty, \quad \textit{for all} \quad \varepsilon > 0 \tag{39}$$

then (37) follows and if $r \geq 1$, $\mathbb{E}\,X = 0$.

In the following part of this subsection we analyze the situation when $\boldsymbol{\alpha}$ is found on the edge of the area $(\frac{1}{2},\infty)^d$. It turns out that if at least one of the coordinates of the vector $\boldsymbol{\alpha}$ is equal to $\frac{1}{2}$, it is enough to be in the zone of attraction of the central limit theorem and to obtain a complete convergence, strong laws of large numbers, B-K theorems – we must improve normalization. This case is explained by the following theorem.

Theorem 3.2.14. ([85], Theorem 3.2) *Let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ be a field of zero-mean ND random variables satisfying WMB condition with r.v. ξ. Suppose, that constants r and α_1 are such that $r \geq 2$, $\alpha_1 = \frac{1}{2}$ and $\alpha_1 r \geq 1$. Then, if*

$$\mathbb{E}\,|\xi|^r\left(\log^+|\xi|\right)^{p-1-\frac{r}{2}} < \infty \quad \textit{and} \quad \mathbb{E}\,\xi^2 = \sigma^2 < \infty, \tag{40}$$

then

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{(r/2)-2}\mathbb{P}\left(|S_{\mathbf{n}}| \geq \sqrt{\prod_{i=1}^{p} n_i \log(\prod_{i=1}^{p} n_i)} \prod_{i=p+1}^{d} n_i^{\alpha_i}\varepsilon\right) < \infty, \tag{41}$$

for $\varepsilon > \sigma_1\sqrt{r-2}$, *where* $p = \max\{k : \alpha_k = \alpha_1\}$ *and* $\sigma_1^2 = \kappa_1\sigma^2$ *(cf. Def. 2.1.4).*

Conversely, suppose that $r = 2$ *and* $p \geq 2$ *or* $r > 2$. *Thus if*

$$\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^{(r/2)-2}\mathbb{P}\left(\max_{\mathbf{k}\preceq\mathbf{n}} |S_{\mathbf{n}}| \geq \sqrt{\prod_{i=1}^{p} n_i \log(\prod_{i=1}^{p} n_i)} \prod_{i=p+1}^{d} n_i^{\alpha_i}\varepsilon\right) < \infty \tag{42}$$

for some $\varepsilon > 0$, *then*

$$\mathbb{E}\,|\xi|^r \left(\log^+|\xi|\right)^{p-1-\frac{r}{2}} < \infty \quad \textit{and} \quad \mathbb{E}\,\xi = 0. \tag{43}$$

The result presented above is a generalization of the theorem established by Gut and Stadtmüller for fields of independent random variables with the same distribution (see [53], Theorem 1.4), to the case of structures with negative dependence and various distributions. The next theorem was obtained directly from Lemma 3.2.10.

Theorem 3.2.15. ([85], Theorem 3.3) *Let* $\{\mathbf{k_n},\, \mathbf{n} \in \mathbb{N}^d\}$ *be a family of elements of* $\mathbb{N}^d$ *such that* $|\mathbf{k_n}|$ *tends to infinity, as* $\max \mathbf{n}$ *tends to infinity;* $\{X_{\mathbf{n},\mathbf{i}}, \mathbf{i} \preceq \mathbf{k_n},\, \mathbf{n} \in \mathbb{N}^d\}$ *be a d-dimensional array, in rows negatively dependent random fields, such that* $\mathbb{E}\,X_{\mathbf{n},\mathbf{i}} = 0$ *and* $\mathbb{E}\,|X_{\mathbf{n},\mathbf{i}}|^r < \infty$ *for* $1 \leq r \leq 2$, $\mathbf{i} \preceq \mathbf{k_n}$ *and* $\mathbf{n} \in \mathbb{N}^d$; *moreover, let* $\{a_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *be a nonnegative real field. Then, if the following conditions are satisfied*

(i) $\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}} \sum_{\mathbf{i}\preceq\mathbf{k_n}} \mathbb{P}(|X_{\mathbf{n},\mathbf{i}}| > \varepsilon) < \infty$, *for all* $\varepsilon > 0$,

(ii) *there exists the constant* $j > 0$ *such that*

$$\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}} \left(\sum_{\mathbf{i}\preceq\mathbf{k_n}} \mathbb{E}\,|X_{\mathbf{n},\mathbf{i}}|^r\right)^j < \infty,$$

then

$$\sum_{\mathbf{n}\in\mathbb{N}^d} a_{\mathbf{n}}\mathbb{P}(|\sum_{\mathbf{i}\preceq\mathbf{k_n}} X_{\mathbf{n},\mathbf{i}}| > \varepsilon) < \infty, \quad \textit{for} \quad \varepsilon > 0.$$

The above theorem is an extension of Sung's results from [114], Dehua and co-authors from [33] obtained for sequences of negatively dependent random variables to the case of random fields with the structure of negative dependence.

At the end of this section, we compare the methods used to obtain the results set out in the second and third sections. Baum-Katz theorems with asymmetric normalization are of course more general than the analogous results obtained in the second section, but they also require additional assumptions that do not weaken in the case of homogeneous normalization. Thus, we can say that the evidence used in both cases: Burkholder and Rosenthal inequalities, and Kahane-Hoffmann-Jørgensen inequality, are adequate to the assertions put forward. The exact argumentation of this statement lies in the proofs of these results.

3.3. Remarks on the Fuk-Nagaev inequality for negatively associated random fields

In this subsection (cf. [85]), we discuss the possibility of generalizing Baum-Katz theorems, obtained for sequences of negatively associated random variables, to the case of negatively associated random fields; in particular, Baum-Katz statements with asymmetric normalization. As in the case of martingale and negative dependence structures, in this case the Fuk-Nagaev inequality is the key. Such a result, for sequences of negatively associated random variables, was proved by Shao and published in [109]. His proof was based on decoupling methods; below we give this comparison theorem.

Let $\{X_i, 1 \leq i \leq n\}$ be a negatively associated sequence and let $\{X_i^*, 1 \leq i \leq n\}$ be a sequence of independent random variables such that X_i^* and X_i have the same distribution for each $i = 1, 2, \ldots, n$. Then

$$Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i\right) \leq Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i^*\right) \tag{44}$$

for any convex and non-decreasing function f on $\mathbf{R}^1$, whenever the expectation on the right side exist.

In the case $d \geq 2$, Bulinski and Suquet (cf, Theorem 2.12 of [14]) have proved, that the comparison theorem does not hold in general for maximum of sums of NA random field.

Theorem 3.3.1. (Shao [109], Theorem 1) *Let $\{X_i, 1 \leq i \leq n\}$ be a negatively associated sequence and let $\{X_i^*, 1 \leq i \leq n\}$ be a sequence of independent random variables such that X_i^* i X_i have the same distribution for each $i = 1, 2, \ldots, n$. Then, for any convex and non-decreasing function f on $\mathbf{R}^1$,*

$$\mathbb{E}f\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i\right) \leq \mathbb{E}f\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i^*\right), \tag{45}$$

whenever the expectation on the right side exist.

In the case $d \geq 2$, Bulinski and Suquet (cf, Theorem 2.12 of [14]) have proved, that the comparison theorem does not hold in general for negatively associated random field.

Theorem 3.3.2. (Bulinski, Suquet [14]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) > f(0)$ (in particular, strictly increasing). Then, for any $d > 1$ there exist a NA random field $X = \{X_{\mathbf{j}},\ \mathbf{j} \in \mathbb{N}^d\}$ and $\mathbf{n}_0 \in \mathbb{N}^d$ such that*

$$\mathbb{E}f\left(\max_{\mathbf{k}\preceq\mathbf{n}_0}\sum_{\mathbf{i}\preceq\mathbf{k}}X_{\mathbf{i}}\right) > \mathbb{E}f\left(\max_{\mathbf{k}\preceq\mathbf{n}_0}\sum_{\mathbf{i}\preceq\mathbf{k}}X_{\mathbf{i}}^*\right), \tag{46}$$

where $X^* = \{X_{\mathbf{j}}^*,\ \mathbf{j} \in \mathbb{N}^d\}$ is decoupled version of X.

The Shao method, in the case of negatively associated random fields, is not only ineffective for this reason. Shao uses the maximum inequality for a non-negative supermartingale. This inequality is not true for random fields, for the same reasons as Doob's maximum inequality (cf. comments on p. 11).

In [85], we show that Kahane-Hoffmann-Jørgensen inequality (20) plays an essential role in proving Baum-Katz theorems. In my opinion, this opens up some new possibilities for the proof of Baum-Katz theorems for random fields with such dependencies, for which we cannot prove the Fuk-Nagaev inequality. In general, the idea is to transfer the problem to other inequalities that are already proven for a given structure or which are easier to prove. The starting point may be, for example, an inequality of the form

$$\mathbb{P}\left(\max_{\mathbf{k}\preceq\mathbf{n}}|S_{\mathbf{k}}|>x\right)\leq\mathbb{P}\left(\max_{\mathbf{k}\preceq\mathbf{n}}|X_{\mathbf{k}}|>\frac{x}{j}\right)+C\left(\frac{1}{x^r}M_{\mathbf{n}}\right)^j, \tag{47}$$

where $M_{\mathbf{n}} := \sum_{\mathbf{k}\preceq\mathbf{n}} \alpha_{\mathbf{k}}$ and $\alpha_{\mathbf{k}} \in \mathbb{R}_+$.

This approach, known as the "general approach to SLLN", was introduced by Fazekas and Klesov in [41].

Chapter 4

Fuk-Nagaev inequalities for fields of martingales and reversed martingales

According to the title, in this chapter we return to the Fuk-Nagaev inequality, presenting the results contained in [77] and [78].

4.1. Fuk-Nagaev inequalities for fields of martingales

What are the differences between the results of section 3.1 and of this one? The Fuk-Nagaev inequality presented in this subsection is valid for the dimension of the set of indices $d = 2$, conditional moments $r \geq 2$ and tail probabilities determined by the real field $\{y_{\mathbf{k}} > 0,\ \mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}\}$.

So let $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}),\ \mathbf{n} \in \mathbb{N}^2\}$ be the field of martingales differences (specified in section 2.2) satisfying the condition $(F4)$ and $\mathbb{E}\,(X_{\mathbf{k}}|\mathcal{G}_{\mathbf{k-1}}) = 0$ a.s. for $\mathbf{k} \preceq \mathbf{n}$ (see notation in section 3.1).

Theorem 4.1.1. *Assume that there exist such positive number fields* $\{b^2_{y_{\mathbf{k}}},\ \mathbf{k} \in \mathbb{N}^2\}$

and $\{a^r_{y_{\mathbf{k}}},\ \mathbf{k}\in\mathbb{N}^2\}$, *that for* $\mathbf{j}\preceq\mathbf{k}\preceq\mathbf{n}$ *we have*

$$\mathbb{E}\,(X^2_{\mathbf{k}}\mathbb{I}(X_{\mathbf{k}}\le y_{\mathbf{k}})|\mathfrak{F}_{\mathbf{j}})\le b^2_{y_{\mathbf{k}}}\ a.s. \tag{48}$$

and

$$\mathbb{E}\,(X^r_{\mathbf{k}}\mathbb{I}(0\le X_{\mathbf{k}}\le y_{\mathbf{k}})|\mathfrak{F}_{\mathbf{j}})\le a^r_{y_{\mathbf{k}}}\ \ a.s. \tag{49}$$

With these assumptions, if

$$\max[t,\ln(\beta x y^{t-1}/A_{t,Y}+1)]>\alpha x y/(e^t B^2_Y), \tag{50}$$

then

$$\begin{aligned}\mathbb{P}(\max_{\mathbf{k}\preceq\mathbf{n}} S_{\mathbf{k}}\ge x)\le&\sum_{\mathbf{k}\preceq\mathbf{n}}\mathbb{P}(X_{\mathbf{k}}\ge y_{\mathbf{k}})\\&+4^{d-1}\exp\left\{\frac{\beta x}{y}-\left(1-\frac{\alpha}{2}\right)\frac{x}{y}\ln\left[\frac{\beta x y^{t-1}}{A_{t,Y}}+1\right]\right\},\end{aligned} \tag{51}$$

where

$$0<\alpha<1,\ \ \beta=1-\alpha,\ \ B^2_Y:=\sum_{\mathbf{k}\preceq\mathbf{n}} b^2_{y_{\mathbf{k}}},\ \ A_{r,Y}:=\sum_{\mathbf{k}\preceq\mathbf{n}} a^r_{y_{\mathbf{k}}}\ \textit{and}\ y\ge\max\{y_{\mathbf{i}},\,\mathbf{i}\preceq\mathbf{n}\};$$

if in condition (9.2.1) we assume an inequality with the opposite direction, then

$$\mathbb{P}(\max_{\mathbf{k}\preceq\mathbf{n}} S_{\mathbf{k}}\ge x)\le\sum_{\mathbf{k}\preceq\mathbf{n}}\mathbb{P}(X_{\mathbf{k}}\ge y_{\mathbf{k}})+4^{d-1}\exp\{-\alpha^2x^2/(2e^rB^2_Y)\}. \tag{52}$$

More inequalities, conclusions and comments are given in [77].

4.2. Fuk-Nagaev inequalities for fields of reversed martingales

The "symmetry" of the definition of a martingale and a reversed martingale translates into many results, here, on the example of one inequality, we show that this is the case of Fuk-Nagaev inequality for random fields.

Let the family of σ-algebras $\{\mathfrak{F}'_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^2\}$ be descending (contracting) with respect to the „$\preceq$ " order in the index set $\mathbb{N}^2$. The $\{(S_{\mathbf{n}}, \mathfrak{F}'_{\mathbf{n}}),\ \mathbf{n} \in \mathbb{N}^2\}$ family is called a reversed martingale random field if it meets the moments and adaptedness conditions such as for the martingale structure and:

$$\mathbb{E}\,(S_{\mathbf{m}}|\mathfrak{F}_{\mathbf{n}}) = S_{\mathbf{n}} \ \text{ a.s., for } \mathbf{n} \succeq \mathbf{m},$$

$$\mathbb{E}\,(\cdot|\mathfrak{F}_{\mathbf{i}}|\mathfrak{F}_{\mathbf{k}}) = \mathbb{E}\,(\cdot|\mathfrak{F}_{\mathbf{i}\vee\mathbf{k}}) \ \text{ a.s.,} \tag{53}$$

where $\mathbf{i}\vee\mathbf{k} := (i_1 \vee k_1, \dots, i_d \vee k_d)$. Condition (53) fulfills the same role as condition (F4) for martingale random fields. Let $\{y_{\mathbf{k}} > 0,\ \mathbf{k} \succeq \mathbf{n}\}$ be a real field and y is such that $y \geq \sup\{y_{\mathbf{k}},\ \mathbf{k} \succeq \mathbf{n}\}\}$, then conditions (9.2.1) and (9.2.1) for $\mathbf{j} \succeq \mathbf{k} \succeq \mathbf{n}$ have the form:

$$\mathbb{E}\,(X_{\mathbf{k}}^2 \mathbb{I}(X_{\mathbf{k}} \leq y_{\mathbf{k}})|\mathfrak{F}'_{\mathbf{j}}) \leq b_{y_{\mathbf{k}}}^2 \ \text{ a.s.} \tag{54}$$

and

$$\mathbb{E}\,(X_{\mathbf{k}}^r \mathbb{I}(0 \leq X_{\mathbf{k}} \leq y_{\mathbf{k}})|\mathfrak{F}'_{\mathbf{j}}) \leq a_{y_{\mathbf{k}}}^r \ \text{ a.s.} \tag{55}$$

So if the condition

$$\mathbb{E}\,(X_{\mathbf{k}} \mathbb{I}(X_{\mathbf{k}} < y)|\mathcal{G}'_{\mathbf{k}+1}) \leq 0 \ \text{ a.s.,} \quad \text{where} \quad \mathcal{G}'_{\mathbf{k+1}} := \bigvee_{\mathbf{j} \not\preceq \mathbf{k}} \mathfrak{F}'_{\mathbf{j}}$$

is satisfied and (53), as well as (54), (55) and (9.2.1), we get the Fuk-Nagaev inequalities in the form of (51) and (52), while the sup and summation take place over the set $\{\mathbf{k} \in \mathbb{N}^2 : \mathbf{k} \succeq \mathbf{n}\}$ (cf. [78]).

Chapter 5

Strong Law of Large Numbers in sector

While considering the strong laws of large numbers for random fields, one more question can be asked, already signaled in the introduction. Suppose that $\mathbf{n}$ – indices of partial sums of $S_{\mathbf{n}}$ are selected from a certain infinite subset of $A \subset \mathbb{N}^d$. When examining the convergence of such sums, we are not considering all of them. Can we suppose (cf. Gabriel, [42]) that thanks to this we will be able to weaken the necessary and sufficient condition for the SLLN. It is one of the most difficult problems of almost sure convergence, connected with the problem of limit theorems for "subsequences" (cf. Klesov, [68] p. 246). The convergence of the random field indexed with the infinite set $A \subset \mathbb{N}^d$ will be understood as follows

$$\xi_{\mathbf{n}} \to 0 \text{ a.s., as } \mathbf{n} \to \infty \text{ (in a given mode of convergence) and } n \in \text{A}. \tag{56}$$

If $\mathbf{n} \to \infty$ is meant by $\max \mathbf{n} \to \infty$ then (56) is equivalent

$$\mathbb{P}\left(|\xi_{\mathbf{n}}| \geq \varepsilon \text{ i.o., } \mathbf{n} \in A\right) = 0, \text{ for all } \varepsilon > 0,$$

where

$$\{B_{\mathbf{n}} \text{ i.o.,} \mathbf{n} \in A\} := \bigcap_{\mathbf{n} \in A} \bigcup_{\mathbf{k} \not\leq \mathbf{n}, \mathbf{k} \in A} B_{\mathbf{k}}.$$

5.1. Results for the random fields with independence structure

To the question posed by Gabriel, the affirmative answer was given by Alan Gut; proving strong laws of large numbers for fields of independent random variables, having the same distribution (cf. [50]); assuming as an infinite subset

$$\mathcal{S}_\theta^d = \left\{ (i_1, ..., i_d) \in \mathbb{N}^d : \theta < \frac{i_l}{i_k} < \frac{1}{\theta}, \text{ for all } l \neq k = 1, ..., d \right\},$$

where θ is a given number from the interval $(0, 1)$. He called the set $\mathcal{S}_\theta^d$ the d-dimensional sector, and the convergence considered – sectoral convergence. Note that

- sectoral convergences in the max and min mode are equivalent;
- since θ, is an arbitrary, the sector does not lose the character of d-dimensionality, and sectoral convergence very well approximates convergence in $\mathbb{N}^d$ in the aspect of statistical applications.

Gut showed that the necessary and sufficient conditions in Marcinkiewicz strong law of large numbers and Hartman-Wintner law of iterated logarithm, for the sectoral convergence of fields of independent random variables with the same distribution, are exactly the same as in the one-dimensional case.

In [81], is generalize the very interesting weighted law of large numbers, obtained by Jajte in [59], to the case of random fields, and in particular of sectoral convergence. More specifically, we provide the necessary and sufficient conditions for convergence

$$\frac{1}{g(|\mathbf{n}|)} \sum_{\mathbf{k} \preceq \mathbf{n}, \mathbf{n} \in \mathcal{S}_\theta^d} \frac{1}{h(|\mathbf{k}|)} \xi_{\mathbf{k}} = \sigma_n \longrightarrow \sigma \quad \text{a.s.,} \quad \text{as} \quad \max \mathbf{n} \to \infty, \tag{57}$$

where $\xi_{\mathbf{k}} = X_{\mathbf{k}} - \mathbb{E}\left(X_{\mathbf{k}} \mathbb{I}[|X_{\mathbf{k}}| \leq \phi(|\mathbf{k}|)]\right)$ and $\phi(y) \equiv g(y)h(y)$.

To unify notations, let us put that $\mathcal{S}_0^d = \mathbb{N}^d$. Therefore, $\mathcal{S}_\theta^d$ is now defined for all $0 \leq \theta < 1$.

Due to the large class of weighting and normalizing functions, the result can be seen as almost sure convergence of (h, g) transforms of random fields or the (h, g) method of the summability of these fields.

In the statement of the main results of this subsection we use functions g, h and $\phi = g \cdot h$ about which we will assume:

(A1) g and h are positive, g is increasing and such that $\lim_{y\to\infty} g(y) = \infty$;

(A2) ϕ is strictly increasing on $\langle t, \infty)$ and $\phi(\langle t, \infty)) = \langle 0, \infty)$, for some $t \geq 0$;

(A3) there exist constants C such that $\phi(y + 1)/\phi(y) \leq C$, for $y \geq k_0$;

(A4) there exist constants a and b such that

$$\begin{aligned} the\phi^2(s) \sum_{k\geq s} \frac{d_\theta(k)}{\phi^2(k)} &\leq as + b, \text{ for } s > t \text{ and } 0 < \theta < 1, \\ \phi^2(s) \sum_{k\geq s} \frac{d_\theta(k)}{\phi^2(k)} &\leq as(\log^+ s)^{d-1} + b, \text{ for } s > t \text{ and } \theta = 0, \end{aligned}$$

where

$$d_\theta(k) = \text{card}\left\{\mathbf{n} \in \mathcal{S}_\theta^d : |\mathbf{n}| = k\right\}, \ k \in \mathbb{N}. \tag{58}$$

In addition, let us put $m_\mathbf{k} := \mathbb{E}\left(X_\mathbf{k}\mathbb{I}[|X_\mathbf{k}| \leq \phi(|\mathbf{k}|)]\right), \ \mathbf{k} \in \mathbb{N}^d$; we can now formulate the previously announced result.

Theorem 5.1.1. ([81], Theorem 1) *Let $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ be a field of independent random variables with the same distribution as the random variable X, and let the functions g, h and ϕ satisfy the assumptions (A1)–(A4), Then the following conditions*

$$\frac{1}{g(|\mathbf{n}|)} \sum_{\mathbf{k}\preceq\mathbf{n}, \mathbf{n}\in\mathcal{S}_\theta^d} \frac{X_\mathbf{k} - m_\mathbf{k}}{h(|\mathbf{k}|)} \longrightarrow 0 \quad \textit{a.s., as } \max \mathbf{n} \to \infty \tag{59}$$

and

$$\mathbb{E}\,\phi^{-1}\left(|X|\right) < \infty, \qquad \textit{for } 0 < \theta < 1, \tag{60}$$
$$\mathbb{E}\,\phi^{-1}\left(|X|\right)\left(\log^{+}\phi^{-1}\left(|X|\right)\right)^{d-1} < \infty, \qquad \textit{for } \theta = 0.$$

are equivalent.

Proof. Let us prove the implication $(59) \implies (60)$. Since the random variables are equidistributed as X we have

$$\frac{m_{\mathbf{k}}}{\phi\left(|\mathbf{k}|\right)} = \frac{E\left(X_{\mathbf{k}} I[|X_{\mathbf{k}}| \leq \phi(|\mathbf{k}|)]\right)}{\phi\left(|\mathbf{k}|\right)} = E\left(\frac{X}{\phi\left(|\mathbf{k}|\right)} I\left[\frac{|X|}{\phi\left(|\mathbf{k}|\right)} \leq 1\right]\right) \to 0, \tag{61}$$

as $\mathbf{k} \to \infty$ in $\mathcal{S}_\theta^r$, by the Lebesgue dominated convergence theorem. Further we may write

$$\frac{X_{\mathbf{n}+\mathbf{1}} - m_{\mathbf{n}+\mathbf{1}}}{\phi\left(|\mathbf{n}+\mathbf{1}|\right)} = \frac{1}{g\left(|\mathbf{n}+\mathbf{1}|\right)}\left(\sum_{\mathbf{a}\in\{0,1\}^r} (-1)^{r+\sum_{i=1}^r a_i} S_{\mathbf{n}+\mathbf{a}}\right), \tag{62}$$

where $S_{\mathbf{n}} = \sum_{\mathbf{k}\preceq\mathbf{n},\mathbf{k}\in\mathcal{S}_\theta^r} \frac{X_{\mathbf{k}}-m_{\mathbf{k}}}{h(|\mathbf{k}|)}$, $\mathbf{1} = (1,...,1)$, $\mathbf{a} =(a_1,...,a_r)$ with $a_i = 0$ or 1. Since g is increasing, for any $\mathbf{a} \in \{0,1\}^r$, we get

$$\left|\frac{1}{g\left(|\mathbf{n}+\mathbf{1}|\right)} (-1)^{r+\sum_{i=1}^r a_i} S_{\mathbf{n}+\mathbf{a}}\right| = \frac{g\left(|\mathbf{n}+\mathbf{a}|\right)}{g\left(|\mathbf{n}+\mathbf{1}|\right)}\left|\frac{S_{\mathbf{n}+\mathbf{a}}}{g\left(|\mathbf{n}+\mathbf{a}|\right)}\right| \leq \tag{63}$$
$$\leq \left|\frac{S_{\mathbf{n}+\mathbf{a}}}{g\left(|\mathbf{n}+\mathbf{a}|\right)}\right| \to 0, \text{ almost surely, as } \mathbf{k} \to \infty \text{ in } \mathcal{S}_\theta^r.$$

Thus from (61), (62) and (63) it follows that

$$\frac{X_{\mathbf{n}}}{\phi\left(|\mathbf{n}|\right)} \to 0, \text{ almost surely, as } \mathbf{n} \to \infty \text{ in } \mathcal{S}_\theta^r$$

and from the Borel–Cantelli lemma we get

$$\sum_{\mathbf{k}\preceq\mathbf{n},\mathbf{k}\in\mathcal{S}_\theta^r} P\left(|X_{\mathbf{n}}| \geq \phi\left(|\mathbf{n}|\right)\right) < \infty, \tag{64}$$

what is equivalent to (60) according to Lemma 2.1 in [48] and Lemma 2.1 in [50].

Now, we prove that $(60) \Longrightarrow (59)$. Let us observe that (64) holds, so that

$$\sum_{\mathbf{k}\in\mathcal{S}_\theta^r} P\left(X_{\mathbf{k}} \neq \overline{X}_{\mathbf{k}}\right) < \infty,$$

thus by the first Borel–Cantelli lemma, it suffices to prove that

$$\frac{1}{g(|\mathbf{n}|)} \sum_{\mathbf{k}\preceq\mathbf{n},\mathbf{k}\in\mathcal{S}_\theta^r} \frac{\overline{X}_{\mathbf{k}} - m_{\mathbf{k}}}{h(|\mathbf{k}|)} \longrightarrow 0, \text{ almost surely, as } \mathbf{n} \to \infty \text{ in } \mathcal{S}_\theta^r.$$

In order to do this, we shall prove that

$$\sum_{\mathbf{k}\in\mathcal{S}_\theta^r} \frac{\mathrm{Var}\left(\overline{X}_{\mathbf{k}} - m_{\mathbf{k}}\right)}{\phi^2\left(|\mathbf{k}|\right)} \leq \sum_{\mathbf{k}\in\mathcal{S}_\theta^r} \frac{E\overline{X}_{\mathbf{k}}^2}{\phi^2\left(|\mathbf{k}|\right)} < \infty. \tag{65}$$

Let us observe that

$$\sum_{\mathbf{k}\in\mathcal{S}_\theta^r} \frac{E\overline{X}_{\mathbf{k}}^2}{\phi^2\left(|\mathbf{k}|\right)} = \sum_{\mathbf{k}\in\mathcal{S}_\theta^r} \frac{E\left(X^2 I[|X| \leq \phi(|\mathbf{k}|)]\right)}{\phi^2\left(|\mathbf{k}|\right)} =$$

$$= EX^2 \sum_{\mathbf{k}\in\mathcal{S}_\theta^r} \frac{I[\phi^{-1}(|X|) \leq |\mathbf{k}|]}{\phi^2\left(|\mathbf{k}|\right)} = EX^2 \sum_{\mathbf{k}\in\mathcal{S}_\theta^r, |\mathbf{k}|\geq\phi^{-1}(|X|)} \frac{1}{\phi^2\left(|\mathbf{k}|\right)} =$$

$$= E\left(\phi\left(\phi^{-1}(|X|)\right)\right)^2 \sum_{k\geq\phi^{-1}(|X|)} \frac{d_\theta(k)}{\phi^2\left(k\right)} < \infty,$$

by A3 and (60). Now the conclusion follows, by (65) and results of Klesov (see [63] and [66]) applied to the random field

$$\left(\frac{\overline{X}_{\mathbf{n}} - m_{\mathbf{n}}}{h(|\mathbf{n}|)}\right)_{\mathbf{n}\in\mathbb{N}^r}.$$

□

Theorem 5.1.1 contains methods of summability such as: Cesàro $(C, 1)$, mean, logarithmic means, Marcinkiewicz's strong law of large numbers transforms, and the like. The methods of summability of random fields have been the subject of many studies;

let us list the most important works and their authors: [57] – Hoffmann-Jørgensen, Su and Taylor, [52] – Gut and Stadtmüller, [100] – Móricz. The strong law of large numbers (59) presented in this subsection complements the results obtained in the aforementioned publications.

5.2. Results for the random fields with dependence structure in pairs

A very important direction of weakening the independence of random variable sequences is to assume the tested condition only in pairs of random variables. Such dependence will be the subject of consideration in this subsection. The names of known probabilists are connected with this research direction. Chung (cf. Theorem 5.2.2 in [26]), assuming pairwise independence, proved the weak law of large numbers. With the same assumption a much more general result was proved by Etemadi in [38], presenting a very "clever" proof of SLLN, without applying the Kolmogorov inequality. Csörgő, Totik and Tandori in [28] and [30] showed that Etemadi's method can be applied to random variables with different distributions and pairwise independent, but they also showed that the assumption cannot be weakened to random variables with uncorrelated pairs. It is also worth mentioning the works of Martikainen [89] as well as Cuesta and Matran [27]. The last dozen or so years have yielded results on sequences of random variables pairwise asymptotically (quasi) independent; one can mention the works of: Chen [24], Gao and co-authors [44], [45] and [46], Li [87], Cheng [25] and pairwise (quasi-asymptotically) dependent, such as the work of Matuła [90] and [91], Azarnosch and co-authors [5]. On the basis of these results, a lot of work on models of investment (financial), insurance (risk), probability(risk) of ruin, etc. was created.

In the following part of this section, based on the results of [82], we will consider random fields with a dependence structure defined on the basis of the concept of the copula; often used in stochastic financial models or actuarial mathematics.

Let us recall (cf. [101]) the definition. The bivariate copula is a function $C : [0,1]^2 \longrightarrow [0,1]$ satisfying the following conditions:

- $C(u,0) = C(0,v) = 0$, $C(u,1) = u$, $C(1,v) = v$,
- $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$.

For any random variables X and Y with the distributions $F_X(x)$ and $F_Y(y)$, there exist copula $C_{X,Y}(u,v)$ such that

$$\mathbb{P}\left(X \leq x, Y \leq y\right) = C_{X,Y}\left(F_X(x), F_Y(y)\right).$$

From Sklar's theorem (cf.[101], Theorem 2.3.3), it is known that the function $C_{X,Y}(u,v)$ is uniquely determined for $(u,v)\ \in$Ran$(F_{X_\mathbf{i}})\ \times$Ran$(F_{Y_\mathbf{j}})$. We will consider the random fields $\left\{X_\mathbf{n},\ \mathbf{n} \in \mathbb{N}^d\right\}$, for which the two-dimensional copula fulfills the condition

$$C_{X_\mathbf{i},X_\mathbf{j}}(u,v) - uv \leq q_{\mathbf{i},\mathbf{j}} uv(1-u)(1-v), \tag{66}$$

where $(u,v)\ \in$Ran$(F_{X_\mathbf{i}})\ \times$Ran$(F_{Y_\mathbf{j}})$ and $q_{\mathbf{i},\mathbf{j}} \geq 0$ for every $\mathbf{i} \neq \mathbf{j}$. Note that condition (66) can be represented in the following equivalent form

$$\begin{aligned} &\mathbb{P}\left(X_\mathbf{i} \leq s, X_\mathbf{j} \leq t\right) - \mathbb{P}\left(X_\mathbf{i} \leq s\right)\mathbb{P}\left(X_\mathbf{j} \leq t\right) \\ &\leq q_{\mathbf{i},\mathbf{j}}\mathbb{P}\left(X_\mathbf{i} \leq s\right)\mathbb{P}\left(X_{\ \mathbf{j}} \leq t\right)\mathbb{P}\left(X_\mathbf{i} > s\right)\mathbb{P}\left(X_\mathbf{j} > t\right) \end{aligned} \tag{67}$$

for every $s,t \in \mathbb{R}$.

The dependence structure defined by condition (67) comprise random fields (sequences of random variables) negatively (quadrant) dependent, (in particular pairwise independent), as well as structures defined by copulas of Farlie-Gumbel-Morgerstern, Ali-Mikhail-Haq and the Plackett families of copulas (cf.[91]). The random fields considered are also related to fields of asymptotically quadrant-independent and asymptotically quadrant sub-independent random variables.

The fundamental role in proof of theorems is provided by a version of the second Borel-Cantelli lemma, for events dependent in pairs with the structure of dependence of the considered random field.

Lemma 5.2.1. ([82], Lemma 3.3) *Let* $\left\{A_{\mathbf{n}} \in \mathfrak{F},\ \mathbf{n} \in \mathcal{S}_\theta^d\right\}$ *be a family of events on some probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and* $\left\{q_{\mathbf{i},\mathbf{j}},\ \mathbf{i},\mathbf{j} \in \mathcal{S}_\theta^d\right\}$ *a field of positive numbers satisfying the following conditions:*

(i) $\sum\limits_{\mathbf{n}\in\mathcal{S}_\theta^d} \mathbb{P}\left(A_{\mathbf{n}}\right) = \infty,$

(ii) there exist $\mathbf{n}_0 \in \mathcal{S}_\theta^d$ *such that for all* $\mathbf{k},\mathbf{j} \in \widehat{\mathcal{S}}_\theta^d(\mathbf{n}_0)$
$\mathbb{P}\left(A_{\mathbf{k}} \cap A_{\mathbf{j}}\right) - \mathbb{P}\left(A_{\mathbf{k}}\right)\mathbb{P}\left(A_{\mathbf{j}}\right) \leq q_{\mathbf{i},\mathbf{j}}\mathbb{P}\left(A_{\mathbf{k}}\right)\mathbb{P}\left(A_{\mathbf{j}}\right),$

(iii) $\sup\limits_{\mathbf{k},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^d(\mathbf{n}_0)} q_{\mathbf{k},\mathbf{j}} < \infty,$

where $\widehat{\mathcal{S}}_\theta^d\left(\mathbf{n}\right) := \mathcal{S}_\theta^d \backslash \left(\mathbf{n}\right)$ *for* $\mathbf{n} \in \mathcal{S}_\theta^d$.
Thus for every $\mathbf{m} \not\leq \mathbf{n}_0$

$$\mathbb{P}\left(A_{\mathbf{n}}, i.o.\ \mathbf{n} \in \mathcal{S}_\theta^d\right) \geq \frac{1}{1 + \sup_{\mathbf{k},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^d(\mathbf{m})} q_{\mathbf{k},\mathbf{j}}}.$$

Proof. Using the idea of Petrov [105] we get

$$\begin{aligned} & P\left(\bigcup_{\mathbf{k}\in\mathcal{S}_\theta^r(\mathbf{n},\mathbf{l})} A_{\mathbf{k}}\right) \\ \geq \ & \left(1 + \sup_{\mathbf{i},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n},\mathbf{l})} q_{\mathbf{i},\mathbf{j}}\right)^{-1} \frac{\left(\sum_{\mathbf{k}\in\mathcal{S}_\theta^r(\mathbf{n},\mathbf{l})} P(A_{\mathbf{k}})\right)^2}{\left(\sum_{\mathbf{k}\in\mathcal{S}_\theta^r(\mathbf{n},\mathbf{l})} P(A_{\mathbf{k}})\right)^2 + \sum_{\mathbf{k}\in\mathcal{S}_\theta^r(\mathbf{n},\mathbf{l})} P(A_{\mathbf{k}})} \end{aligned}$$

thus taking the limit over $\mathbf{l}$ of the both sides, we obtain

$$P\left(\bigcup_{\mathbf{k}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n})} A_{\mathbf{k}}\right) \geq \frac{1}{1 + \sup_{\mathbf{i},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n})} q_{\mathbf{ij}}}.$$

Now, one can easily see that

$$\bigcap_{\mathbf{n}\in\mathcal{S}_\theta^r} \bigcup_{\mathbf{k}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n})} A_{\mathbf{k}} = \bigcap_{N=1}^{\infty} \bigcup_{\mathbf{k}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_{\mathbf{k}}$$

where $\mathbf{N} = (N, ..., N)$. The sequence of events $B_N = \bigcup_{\mathbf{k}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_\mathbf{k}$ is decreasing, hence

$$P\left(A_\mathbf{n}, \text{i.o. } \mathbf{n} \in \mathcal{S}_\theta^r\right) = \lim_{N\to\infty} P\left(\bigcup_{\mathbf{k}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_\mathbf{k}\right) \geq \frac{1}{1+\sup_{\mathbf{i},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n}_0)} q_{\mathbf{i},\mathbf{j}}}.$$

□

Remark 1. *Under conditions of Lemma 3.3 with (iii) replaced by*

(iii') $\sup_{\mathbf{k},\mathbf{j}\in\widehat{\mathcal{S}}_\theta^r(\mathbf{n})} q_{\mathbf{k},\mathbf{j}} \to 0$, *as* $\mathbf{n}\to\infty$,

we have $P\left(A_\mathbf{n}, i.o.\ \mathbf{n} \in \mathcal{S}_\theta^r\right) = 1$

Lemma (5.2.1) allows to prove the following two statements. The first of these can be compared with the result contained in [69], where the author deals with the class of random fields with an asymptotically quadrant sub-independent structure, i.e. a narrower class of random fields than we consider. Additionally, it assumes that the field of coefficients affecting dependence in this structure – $q_{\mathbf{k},\mathbf{j}}$, depends on $\max|\mathbf{k}-\mathbf{j}|$; we do not make any assumptions of this type. In addition, we provide necessary and sufficient conditions; in [69] we only find a sufficient condition. For the sake of brevity, to formulate this theorem, let us introduce the necessary notations:

$$\mathcal{S}_\theta^d(\mathbf{n}) := \mathcal{S}_\theta^d \cap (\mathbf{n}), \qquad \mathcal{S}_\theta^d(|\mathbf{n}|) := \left\{\mathbf{i}\in\mathcal{S}_\theta^d : |\mathbf{i}| \leq |\mathbf{n}|\right\}.$$

Theorem 5.2.2. ([82], Theorem 2.1) *Let* $\left\{X_\mathbf{n},\ \mathbf{n}\in\mathcal{S}_\theta^d\right\}$ *be a field of equidistributed random variables with dependence structure defined by condition (66) with* $q_{\mathbf{i},\mathbf{j}}$ *such that*

$$\sum_{\mathbf{j}\in\mathcal{S}_\theta^d}\ \sum_{\mathbf{i}\in\mathcal{S}_\theta^d(|\mathbf{j}|),\mathbf{i}\neq\mathbf{j}} |\mathbf{j}|^{-2} q_{\mathbf{i},\mathbf{j}} < \infty \text{ and } \sup_{\mathbf{i},\mathbf{j}\in\mathcal{S}_\theta^d} q_{\mathbf{i},\mathbf{j}} < \infty. \tag{68}$$

Thus, the following conditions are equivalent:

$$(i) \qquad \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}\in\mathcal{S}_\theta^d(\mathbf{n})} (X_\mathbf{k} - m_\mathbf{k}) \to 0 \quad a.s., \text{ as } \max\mathbf{n}\to\infty, \tag{69}$$

$$(ii) \qquad \mathbb{E}\,|X| \left(\log^+ |X|\right)^{d-1} < \infty, \qquad \textit{if}\ \theta = 0, \tag{70}$$
$$\mathbb{E}\,|X| < \infty, \qquad \textit{if}\ \theta \in (0,1).$$

Proof. For $a < b$ let us define $\varphi_{a,b}(t) = aI[t \le a] + tI[a < t < b] + bI[t \ge b]$. Let us begin with the sufficient condition (70)$\Longrightarrow$(69). Obviously

$$\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} P\left(\varphi_{-|\mathbf{j}|,|\mathbf{j}|}(X_\mathbf{j}) \neq X_\mathbf{j}\right) = \sum_{\mathbf{j} \in \mathcal{S}_\theta^r} P\left(|X_\mathbf{j}| > |\mathbf{j}|\right) \tag{71}$$

and the r.h.s. of (71) is finite by (70). Therefore it is enough to prove (69) for truncated random field $\left\{\varphi_{-|\mathbf{j}|,|\mathbf{j}|}(X_\mathbf{j}), \mathbf{j} \in \mathcal{S}_\theta^r\right\}$. To be exact, we shall prove it for $X'_\mathbf{j} = \varphi^+_{-|\mathbf{j}|,|\mathbf{j}|}(X_\mathbf{j}) = \varphi_{0,|\mathbf{j}|}(X_\mathbf{j})$ and $X''_\mathbf{j} = \varphi^-_{-|\mathbf{j}|,|\mathbf{j}|}(X_\mathbf{j}) = -\varphi_{-|\mathbf{j}|,\mathbf{0}}(X_\mathbf{j})$. Now, let us observe that the family $\left\{X'_\mathbf{j}, \mathbf{j} \in \mathcal{S}_\theta^r\right\}$ satisfies the assumptions (i)-(iii) of Lemma 3.1. To check (iv), let us at first note, that by our assumption (66) and Lemma 3.2 we get for $\mathbf{j} \neq \mathbf{k}$

$$\begin{aligned}
\operatorname{Cov}\left(X'_\mathbf{j}, X'_\mathbf{k}\right) &= \int_0^{|\mathbf{j}|}\int_0^{|\mathbf{k}|} \left[P\left(X_\mathbf{j} \le u, X_\mathbf{k} \le v\right) - P\left(X_\mathbf{j} \le u\right) P\left(X_\mathbf{k} \le v\right)\right] du dv \\
&\le q_{\mathbf{i},\mathbf{j}} \int_0^{|\mathbf{j}|} P\left(X_\mathbf{j} > u\right) du \int_0^{|\mathbf{k}|} P\left(X_\mathbf{k} > v\right) dv = q_{\mathbf{j},\mathbf{k}} E X'_\mathbf{j} E X'_\mathbf{k} \\
&\le q_{\mathbf{j},\mathbf{k}} \left(E|X_\mathbf{1}|\right)^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} \sum_{\mathbf{j} \in \mathcal{S}_\theta^r(|\mathbf{k}|)} |\mathbf{k}|^{-2}\, Cov^+\left(X_\mathbf{j} I\left[X_\mathbf{j} \le |\mathbf{j}|\right], X_\mathbf{k} I\left[X_\mathbf{k} \le |\mathbf{k}|\right]\right) \\
\le\ & \left(E|X_\mathbf{1}|\right)^2 \sum_{\mathbf{j} \in \mathcal{S}_\theta^r} \sum_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|), \mathbf{i} \neq \mathbf{j}} |\mathbf{j}|^{-2} q_{\mathbf{i},\mathbf{j}} + \sum_{\mathbf{j} \in \mathcal{S}_\theta^r} |\mathbf{j}|^{-2} \operatorname{Var}\left(X'_\mathbf{j}\right) < \infty
\end{aligned}$$

by (68) and since $\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} |\mathbf{j}|^{-2} \operatorname{Var}\left(X'_\mathbf{j}\right)$ is bounded by $E\,|X_\mathbf{1}| \left(\log_+ |X_\mathbf{1}|^{r-1}\right)$, if $\theta = 0$ and by $E\,|X_\mathbf{1}|$ in the sectorial case $\theta \in (0,1)$ (cf. [100]). Thus, by Lemma 3.1 we have

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} \left(X'_\mathbf{k} - E X'_\mathbf{1}\right) \to 0, \text{ almost surely as } \mathbf{n} \to \infty \text{ in } \mathcal{S}_\theta^r$$

similar result holds for $\left\{X''_{\mathbf{j}}, \mathbf{j} \in \mathcal{S}_\theta^r\right\}$, furthermore $|\mathbf{n}|P(|X_1| > |\mathbf{n}|) \to 0$ as $\mathbf{n} \to \infty$ and the proof of sufficiency is completed.

To prove necessity (69)$\Longrightarrow$(70), let us observe that by the standard arguments we get $X_{\mathbf{n}}/|\mathbf{n}| \to 0$ almost surely as $\mathbf{n} \to \infty$ in $\mathcal{S}_\theta^r$. Thus, by Lemma 3.3,

$$\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} P(|X_{\mathbf{n}}| \geq |\mathbf{n}|) < \infty,$$

what gives (70). □

The more classic version of the SLLN for fields of pairwise dependent random variables that meet the condition (66) can be formulated in the following way.

Theorem 5.2.3. ([82], Theorem 2.2) *Let $\left\{X_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_\theta^d\right\}$ be a field of equidistributed random variables with dependence structure satisfying condition (66) with condition (68).Then, the condition (70) is equivalent to – there exist the constant c, such that*

$$\lim(\max)\frac{1}{M_\theta(\mathbf{n})}\sum_{\mathbf{k} \in \mathcal{S}_\theta^d(\mathbf{n})} X_{\mathbf{k}} \to c, \ \textit{a.s.}, \tag{72}$$

where $M_\theta(\mathbf{n}) := Card(\mathcal{S}_\theta^d(\mathbf{n}))$. If (70) holds, then $c = \mathbb{E}X$.

Many authors dealing with this type of theorems focused on the generalizations of the infinite subset $A \subset \mathbb{N}^d$. We can find examples of such results in the works of Bass and Pyke [7], [8], Klesov [68] (cf. Theorem 9.8 generalizing the edge of the sector to a functional form), Klesov and Rychlik [67], Indlekofer and Klesov [60]. In all these works, Kolmogorov strong laws of large numbers were obtained for fields of independent random variables with the same distribution. In our research, we conclude that the sector is sufficient for statistical applications. We focus only on the generalization of the random field structure and/or on generalization of the strong laws of large numbers for sectoral convergence.

Chapter 6

Inequalities used in proofs of SLLN for fields with values in Banach spaces

In this part of the outline, we will present some inequalities that allow to prove the strong laws of large numbers for fields of random elements taking values in the Banach space. We will also give the characterization of the geometry of such Banach spaces in which the SLLN occurs.

6.1. Hajek-Rényi-Chow inequality

In this subsection, we present results from [86]. As we said in the introduction, Doob's classic inequality cannot be generalized to random fields, and thus to Hajek-Rényi-Chow inequalities (H-R-C inequality). Christofides and Serfling in [23] proved a certain version of H-R-C inequality for a martingale random fields. Analyzing the proof of Theorem 2.2 from the above-mentioned work [23], we noticed that the conclusion can be extended to random fields with sub-martingale structure.

Theorem 6.1.1. ([86], Theorem 1.1) *Let* $\{(Y_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}}),\ \mathbf{k} \in \mathbb{N}^d\}$ *be a sub-martingale random field; filtration* $\{\mathfrak{F}_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\}$ *satisfies condition* $(F4)$ *and let* $\{C_{\mathbf{k}},\, \mathbf{k} \in \mathbb{N}^d\}$ *be a non-increasing, real field. Then for all* $\lambda > 0$ *have*

$$\lambda \mathbb{P}\left(\sup_{\mathbf{k}\not\preceq\mathbf{m},\mathbf{k}\preceq\mathbf{n}} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda\right) \leq \min_{1\leq s\leq r}\left\{\sum_{\mathbf{k}\not\preceq\mathbf{m},\mathbf{k}\preceq\mathbf{n}} (C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1})\mathbb{E}\, Y_{\mathbf{k}}^+ \right.$$
$$\left. - \sum_{k_i, i\neq s} C_{\mathbf{k};s;n_s} \int_{[\bigcup_{k_s=1}^{n_s} B^{(s)}_{k_1,\ldots,k_r}]^c} Y^+_{\mathbf{k};s;n_s}\, d\mathbb{P}\right\} \tag{73}$$
$$\leq \min_{1\leq s\leq r}\left\{\sum_{\mathbf{k}\not\preceq\mathbf{m},\mathbf{k}\preceq\mathbf{n}} (C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1})\mathbb{E}\, Y_{\mathbf{k}}^+\right\},$$

where $C_{\mathbf{k};s;\alpha} = C_{k_1,\ldots,k_{s-1},\alpha,k_{s+1},\ldots,k_r}$; *if* $k_i > n_i$ *for some* $i = 1, 2, \ldots, d$ *thus we put* $C_k = 0$.

Proof. In the multidimensional martingale case, Theorem 6.1.1 was proved by Christofides and Serfling [23] using properties of submartingale fields (see also Remark 1), where *sup* was taken over set $\{\mathbf{k} \in \mathbb{N} : \mathbf{k} \preceq \mathbf{n}\}$ thus it's enough to prove that assertion of the above theorem holds true for more general sets

$$D = \{\mathbf{k} \in \mathbb{N} : \mathbf{k} \not\preceq \mathbf{m}, \mathbf{k} \preceq \mathbf{n}\}.$$

Assume without loss of generality, that the sum on right-hand side of (4) has minimum for $s_0 \neq 1$. Let us define the disjoint partition of D as follow :

$$D_1 = \{\mathbf{j} = (j_1, \ldots, j_r) : m_1 + 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq m_2, \ldots, 1 \leq j_r \leq m_r\},$$
$$D_i = \{\mathbf{j} = (j_1, \ldots, j_r) : 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, \ldots, m_i + 1 \leq j_i \leq n_i,$$
$$1 \leq j_{i+1} \leq m_{i+1}, \ldots, 1 \leq j_r \leq m_r\}$$

for $i = 2, 3, \ldots, r$.
It is easy to see, that $\bigcup_{i=1}^r D_i = D$ and $D_i \cap D_j = \emptyset$ for $i \neq j$.

Now, let us observe that we can apply Theorem 1 to the "cubes" $\{\mathbf{k} \in \mathbb{N}^{\mathbf{d}} : \mathbf{l} \preceq \mathbf{k} \preceq \mathbf{n}\}$, where $\mathbf{1} \leq \mathbf{l} < \mathbf{n}$.
Thus we have

$$\lambda P(\sup_{\mathbf{k}\in D} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda) = P(\bigcup_{\mathbf{k}\in D} [C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda]) =$$
$$P(\bigcup_{i=1}^{r} \bigcup_{\mathbf{k}\in D_i} [C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda]) \leq \sum_{i=1}^{r} P(\bigcup_{\mathbf{k}\in D_i} [C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda]) \leq$$
$$\sum_{i=1}^{r} P(\sup_{\mathbf{k}\in D_i} C_{\mathbf{k}} Y_{\mathbf{k}} \geq \lambda) \leq \sum_{i=1}^{r} \min_{1\leq s\leq r} \sum_{\mathbf{k}\in D_i} \{(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}) EY_{\mathbf{k}}^{+}\} \leq \tag{74}$$
$$\min_{1\leq s\leq r} \sum_{i=1}^{r} \sum_{\mathbf{k}\in D_i} \{(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}) EY_{\mathbf{k}}^{+}\}$$
$$\leq \min_{1\leq s\leq r} \sum_{\mathbf{k}\in D} \{(C_{\mathbf{k}} - C_{\mathbf{k};s;k_s+1}) EY_{\mathbf{k}}^{+}\}$$

□

Remark 2. ([86]) *In the proof of theorem 2.2 of [23], authors construct, the sets* $B_{\mathbf{k}}^{(i)}$ *and say: "An explicit expression of the sets* $B_{\mathbf{k}}^{(i)}$ *in terms of the sets* $A_{\mathbf{k}}$ *is possible to derive , but such formula is notationally messy and complicated." It seems, that in the proof of Theorem 1, we can use the sets* $\tilde{B}_{\mathbf{k}}^{(i)}$ *constructed as follows (in the case* $r = 2$*, for simplicity)*
Let $\mathbf{n} = (n_1, n_2)$*, set* $Z_i(\omega) = \sup_{1\leq j\leq n_2} C_{ij} Y_{ij}(\omega)$,

$$I^{(1)}(\omega) = \inf_{1\leq i\leq n_1} \{i : Z_i(\omega) \geq \lambda\} \textit{ (or } n_1 + 1 \textit{ if no such i exists)}$$

$$J^{(1)}(\omega) = \inf_{1\leq j\leq n_2} \{j : C_{Ij} Y_{Ij}(\omega) \geq \lambda\} \quad \textit{and set} \quad \tilde{B}_{ij}^{(1)} = \{I^{(1)}(\omega) = i, J^{(1)}(\omega) = j\}$$

The sets $\tilde{B}_{\mathbf{k}}^{(2)}$ *we obtain by changing the order of taking maximum. In this construction we used idea introduced by Zimmerman* [129]. *Similarly to the sets constructed* *by Christofides and Serfling* $\tilde{B}^{(1)}_{\mathbf{k}}$,$\tilde{B}^{(2)}_{\mathbf{k}}$ *are disjoint,* $\mathfrak{F}_{in_2}$ *and* $\mathfrak{F}_{n_1 j}$ *respectively measurable and*

$$\bigcup_{1\le i\le n_1, 1\le j\le n_2} \tilde{B}^{(1)}_{ij} = \bigcup_{1\le i\le n_1, 1\le j\le n_2} \tilde{B}^{(2)}_{ij}$$

$$= \left[\sup_{1\le i\le n_1, 1\le j\le n_2} C_{ij} Y_{ij} \ge \lambda \right]$$

Such construction gives a simple formula and is very intuitive.

Theorem 6.1.1, despite its imperfections, works quite well. With possible minor additional assumptions, we can obtain inequalities for random fields: Hajek-Réni, Kolmogorov, general form of Chow's strong law of large numbers (cf. [23]). Maintaining the assumptions of the previously quoted Serfling and Christofides Theorem 2.2, we modified the event, whose probability is estimated in the inequality of H-R-C. Thanks to this, we have obtained a tool for studying almost sure convergence in the max mode, and not only convergence in the min mode, as was the case in [23]. Therefore, all the results obtained by them remain true also for convergence in the max mode, without additional assumptions. The extension of inequality (73) to a case involving sub-martingale random field allows to apply it to almost sure convergence of random fields with values in the Banach space $(\mathbb{B}, \|\cdot\|)$, when we know that the norm of partial sums $\{\|S_k\|, \mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ are the (real) random field with sub-martingale structure.

Before we proceed to discussing further theorems, we will give some auxiliary results obtained in [86]. The first result is a multidimensional version of Kronecker's lemma, equivalent to the Martikainen version (see [88], p. 435), but a form necessary for proofs of almost sure convergence in the max mode. Note that for $d > 1$ the assumption that the elements of the array $\{x_{(\mathbf{l},\mathbf{m})}, (\mathbf{l}, \mathbf{m}) \in \mathbb{N}^d\}$ are positive, is necessary – by contrast to the one-dimensional case.

Lemma 6.1.2. ([86], Lemma 2.2) *Let* $s, t \ge 1$ *be natural numbers such that* $s + t = d$, $\{a_{\mathbf{l}}, \mathbf{l} \in \mathbb{N}^s\}$ *and* $\{b_{\mathbf{m}}, \mathbf{m} \in \mathbb{N}^t\}$ *fields of increasing (with respect to* $|\mathbf{l}|$ *and*

$|\mathbf{m}|$ *respectively), positive numbers such that* $a_{\mathbf{l}} \to \infty$, $b_{\mathbf{m}} \to \infty$ *in* max *mode. Furthermore, let* $\{x_{(\mathbf{l},\mathbf{m})}, (\mathbf{l},\mathbf{m}) \in \mathbb{N}^d\}$ *be an array of positive numbers satisfying inequality*

$$\sum_{(\mathbf{l},\mathbf{m})\in\mathbb{N}^d} \frac{x_{(\mathbf{l},\mathbf{m})}}{a_{\mathbf{l}} b_{\mathbf{m}}} < \infty.$$

Thus

$$\frac{1}{a_{\mathbf{N}_1}} \sum_{(\mathbf{l},\mathbf{m})\preceq(\mathbf{N}_1,\mathbf{N}_2)} \frac{x_{(\mathbf{l},\mathbf{m})}}{b_{\mathbf{m}}} \to 0, \quad as \quad \max \mathbf{N}_1 \to \infty,$$

for every $\mathbf{N}_2 \in \mathbb{N}^t$.

The next lemma gives the characterization of the geometry of the Banach space in terms of Marcinkiewicz-Zygmund inequalities for fields of random elements with the structure of independence. The result is a conclusion from Theorem 2.1 obtained by Woyczyński in [125].

In this subsection, $\mathbb{E}\, X$ will mean the Bochner integral of the random element $X(\omega)$ with values in the Banach space $(\mathbb{B}, \|\cdot\|)$.

Lemma 6.1.3. ([86], Lemma 2.3) *Let* $(\mathbb{B}, \|\cdot\|)$ *be a real separable Banach space, then for* $p, q \in \mathbb{R}$ *such that* $1 \le p \le 2$ *i* $q \ge 1$, *the following conditions are equivalent:*

(i) $\mathbb{B}$ *is R type*$-p$*;*

(ii) There exist positive constant C *depend only on* p *and* q *such that for every field* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ *of independent random vectors in* $\mathbb{B}$, *zero-mean with an* $q-$*absolute moment, the following inequality*

$$\mathbb{E}\, \| \sum_{\mathbf{k}\preceq\mathbf{n}} X_k \|^q \le C\mathbb{E}\, (\sum_{\mathbf{k}\preceq\mathbf{n}} \|X_{\mathbf{n}}\|^p)^{q/p} \tag{75}$$

holds.

Theorem 6.1.1, along with Lemmas 6.1.2 and 6.1.3, allows to prove the following Brunk-Prokhorov law of large numbers, being a partial generalization of Theorem 3.1 proved by Woyczyński in [125].

Brunk-Prokhorov SLLN in R-type Banach space

Theorem 6.1.4. ([86], Theorem 2.4) *Let* $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ *be field of independent, zero-mean* $\mathbb{B}-$*valued random vectors such that* $\mathbb{E}\,\|X_{\mathbf{n}}\|^{pq} < \infty$ *for every* $\mathbf{n} \in \mathbb{N}^d$*, and*

$$\min_{1\le s\le d} \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbb{E}\,\|X_{\mathbf{n}}\|^{pq}}{n_s|\mathbf{n}|^{pq-q}} < \infty, \ \ \textit{for}\ \ 1 \le p \le 2,\ q > 1. \tag{76}$$

Moreover, if $\mathbb{B}$ *is* $R-$*type p, then*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|} \to 0 \quad \textit{a.s.,} \qquad \textit{as} \quad \max \mathbf{n} \to \infty. \tag{77}$$

Proof. Let $\mathfrak{F}_{\mathbf{n}} = \sigma(X_{\mathbf{k}}, \mathbf{k} \le \mathbf{n})$, since $X_{\mathbf{k}}$ are independent, $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ satisfy (1) and $\{\|S_{\mathbf{k}}\|^{pq}, \mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ is real, nonnegative submartingale. By definition of max mode convergence of elements of **B** and "event occurs finitely (infinitely) often" it is enough to prove

$$\lim_{N\to\infty} P(\sup_{(k_1,k_2)\not\le(N,N)} \frac{\|S_{(k_1,k_2)}\|}{k_1k_2} \ge \lambda) = 0 \tag{78}$$

for any $\lambda > 0$. Let us observe, that by Remark 1, Lemma 1 and Hölder inequality, we

get

$$\begin{aligned}
&\lambda^{pq} P(\sup_{\mathbf{k}\not\le\mathbf{N}} \frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \ge \lambda) \\
&\le \min_{1\le s\le r} \sum_{\mathbf{k}\not\le\mathbf{N}} \left(\frac{1}{|\mathbf{k}|^{pq}} - \frac{1}{(|\mathbf{k}|^{pq}; s; k_s+1)} \right) E\|S_{\mathbf{k}}\|^{pq} \\
&\le C \min_{1\le s\le r} \sum_{\mathbf{k}\not\le\mathbf{N}} \left(\frac{1}{|\mathbf{k}|^{pq}} - \frac{1}{(\mathbf{k}^{pq}; s; k_s+1)} \right) |\mathbf{k}|^{q-1} \sum_{\mathbf{j}\le\mathbf{k}} E\|X_{\mathbf{j}}\|^{pq} \\
&= C \sum_{\mathbf{k}\not\le\mathbf{N}} \frac{1}{k_{s_0}|\mathbf{k}|^{pq-q+1}} \sum_{\mathbf{j}\le\mathbf{k}} E\|X_{\mathbf{j}}\|^{pq}
\end{aligned} \tag{79}$$

On the other hand, for some constant $C > 0$, we have

$$
\begin{aligned}
&\sum_{(k_1,k_2)\preceq(N,N)} \frac{(k_1k_2)^{q-1}}{k_1^{pq}k_2^{pq+1}} \sum_{(j_1,j_2)\preceq(k_1,k_2)} E\|X_{j_1j_2}\|^{pq} = \\
&= \sum_{(k_1,k_2)\preceq(N,N)} E\|X_{k_1k_2}\|^{pq} \sum_{(k_1,k_2)\preceq(j_1,j_2)\preceq(N,N)} \frac{(j_1j_2)^{q-1}}{j_1^{pq}j_2^{pq+1}} \\
&\preceq C \sum_{(k_1,k_2)\preceq(N,N)} E\|X_{k_1k_2}\|^{pq} \left(\frac{1}{N^{2pq-2q+1}} + \frac{1}{k_1^{pq-q}N^{pq-q+1}} \right. \\
&\left. + \frac{1}{N^{pq-q}k_2^{pq-q+1}} + \frac{1}{k_1^{pq-q}k_2^{pq-q+1}} \right).
\end{aligned}
\tag{80}
$$

By, the multidimensional version of Kronecker Lemma (see Martikainen [10] p.435 or Lemma 2.5 of Su.K.-L.[17]) and assumption

$$\sum_{(k_1,k_2)\preceq(N,N)} \frac{E\|X_{k_1k_2}\|^{pq}}{N^{2pq-2q+1}} \to 0 \quad \text{as} \quad N\to\infty$$

Furthermore, by the classical Kronecker Lemma

$$\sum_{(k_1,k_2)\preceq(N,N)} \frac{E\|X_{k_1k_2}\|^{pq}}{k_1^{pq-q}N^{pq-q+1}} \to 0 \quad \text{as} \quad N\to\infty$$

as well as

$$\sum_{(k_1,k_2)\preceq(N,N)} \frac{E\|X_{k_1k_2}\|^{pq}}{N^{pq-q}k_2^{pq-q+1}} \to 0 \quad \text{as} \quad N\to\infty.$$

Hence, the last sum in (9) is at most equal than

$$C \sum_{(k_1,k_2)\preceq(N,N)} \frac{k_1E\|X_{k_1k_2}\|^{pq}}{(k_1k_2)^{pq-q+1}}$$

for some constant C and sufficiently large N .
Similarly, we can prove, that

$$\sum_{(k_1,k_2)\preceq(N,N)} \frac{(k_1k_2)^{q-1}}{k_1^{pq+1}k_2^{pq}} \sum_{(j_1,j_2)\preceq(k_1,k_2)} E\|X_{j_1j_2}\|^{pq} \preceq C \sum_{(k_1,k_2)\preceq(N,N)} \frac{k_2E\|X_{k_1k_2}\|^{pq}}{(k_1k_2)^{pq-q+1}}. \tag{81}$$

Thus by assumption (5) one of the series (9) or (10) is convergent, hence the term on the right-hand side of (8) tend to 0 as $N \to \infty$, what proves (7) and finally Theorem 6.1.4. □

Corollary 6.1.5. ([86], Corollary 1) *If in Theorem 2 we replace condition (5) by the following*

$$\sum_{\boldsymbol{n}\in\mathbb{N}^d} \frac{E\|X_{\mathbf{n}}\|^{pq}}{|\mathbf{n}|^{pq-q+1}} < \infty \tag{82}$$

and the rest of the assumptions remain valid, then the assertion is also true, it is SLLN (6) holds.

Proof. Let

$$\xi_{n_1,\dots,n_r,1} = X_{n_1,\dots,n_r,} = X_{\mathbf{n}} \quad \text{and}$$

$$\xi_{\widetilde{\mathbf{n}}} = \xi_{n_1,\dots,n_r,n_{r+1}} = 0 \quad \text{for} \quad n_{r+1} > 1, \quad \widetilde{\mathbf{n}} \in N^{r+1}.$$

Of course

$$T_{\widetilde{\mathbf{n}}} = \sum_{\widetilde{\mathbf{k}}\preceq\widetilde{\mathbf{n}}} \xi_{\widetilde{\mathbf{k}}} = \sum_{\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} = S_{\mathbf{n}} \quad \text{and}$$

$$\min_{1\le s\le r+1}\sum_{\widetilde{\mathbf{k}}\in N^{r+1}}\frac{k_s E\|\xi_{k_1,\dots,k_s,\dots,k_r,k_{r+1}}\|^{pq}}{|k_1\cdot\ldots\cdot k_r\cdot k_{r+1}|^{pq-q+1}}$$
$$=\min_{1\le s\le r+1}\sum_{\mathbf{k}\in\mathbb{N}^d}\frac{k_s E\|\xi_{k_1,\dots,k_s,\dots,k_r,1}\|^{pq}}{|\mathbf{k}|^{pq-q+1}} \tag{83}$$
$$=\sum_{\mathbf{k}\in\mathbb{N}^d}\frac{E\|X_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq-q+1}}<\infty.$$

where $\widetilde{\mathbf{n}}=(n_1,\dots,n_d,n_{d+1})\in N^{d+1}$ and $\max\widetilde{\mathbf{n}}\to\infty$, then $T_{\widetilde{\mathbf{n}}}/|\widetilde{\mathbf{n}}|\to 0$ a.s. This implies

$$S_{\mathbf{n}}/|\mathbf{n}|\to 0 \qquad \text{a.s.}\quad \text{as}\quad \max\mathbf{n}\to\infty.$$

□

The relations between the geometry of the Banach space, the rate of convergence in the weak law of large numbers (WLLN) and the Brunk-Prokhorov law of large numbers is given by the following theorem, which is the equivalent to random elements sequences case (cf. [125], Theorem 3.2).

Theorem 6.1.6. ([86], Theorem 2.7) *Let p and q be a real numbers such that $1\le p\le 2$ and $q\ge 1$, then the following conditions are equivalent:*

(i) $\mathbb{B}$ *is of R-type p;*

(ii) for every $\lambda>0$ there exists the constant C_λ such that for any field $\{X_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$ of independent random vectors taking values in $\mathbb{B}$, the following inequality

$$\sum_{\mathbf{n}\in\mathbb{N}^d}|\mathbf{n}|^{-1}\mathbb{P}\left(\frac{\|S_{\mathbf{n}}\|}{|\mathbf{n}|}\ge\lambda\right)\le C_\lambda\sum_{\mathbf{n}\in\mathbb{N}^d}\frac{\mathbb{E}\,\|X_{\mathbf{n}}\|^{pq}}{|\mathbf{n}|^{pq-q+1}}$$

is satisfied

For r=1, theorem due to Woyczynski [125]. Combining above Theorem 6.1.6 with the result of Rosalsky and Than [107], Theorem 3.1 we get the following corollary.

Corollary 6.1.7. ([86], Corollary 2) *Let* $1 \le p \le 2$, $q \ge 1$ *and* $\mathbb{B}$ *be a separable Banach space. If* $\{X_{\mathbf{k}},\ \mathbf{k} \in \mathbb{N}^d\}$ *are family of independent,* $\mathbb{B}$ *-valued, zero-mean random vectors, then the following conditions are equivalent:*

(i) For every $q \ge 1$ *and* $\lambda > 0$ *there exists* C_λ *such that for any vectors* $\{X_{\mathbf{k}},\ \mathbf{k} \in \mathbb{N}^d\}$

$$\sum_{\mathbf{k}\in\mathbb{N}^d} |\mathbf{k}|^{-1} P\left(\frac{\|S_{\mathbf{k}}\|}{|\mathbf{k}|} \ge \lambda\right) \le C_\lambda \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{E\|X_{\mathbf{k}}\|^{pq}}{|\mathbf{k}|^{pq-q+1}},$$

(ii) for every random vectors $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ *the condition*

$$\sum_{\mathbf{k}\in\mathbb{N}^d} \frac{E\|X_{\mathbf{k}}\|^{p}}{|\mathbf{k}|^{p}} < \infty$$

implies that the SLLN holds.

The idea presented, based on Theorem 6.1.1, can be successfully used to study almost sure convergence of random fields with values in the Banach space with a martingale dependence structure; this of course requires stronger assumptions for the geometry of space – p-uniform smoothness (or p-smoothness). For example, in this way Brunk-Prokhorov strong law of large numbers was proven in [113] for the fields of martingale differences (along with the characterization of the geometry of the Banach space).

6.2. Marcinkiewicz type inequality

In this part of the section we give a certain version of Marcinkiewicz's inequality for random fields, which allows us to receive the Brunk-Prokhorov strong laws of large numbers. Application of this inequality in the proofs of SLLN, under assumption of the truth of the weak law of large numbers, does not require information about the geometry of the Banach space. Such claims are a good complement to the results of the previous subsection.

Lemma 6.2.1. ([86], Lemma 2.11) *Let* $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *be a field of independent random vectors taking values in separable Banach space* $\mathbb{B}$ *and* q *be the real number, thus the following two statements:*

(i) if $1 \le q \le 2$ *then*

$$\mathbb{E}\,|\|S_{\mathbf{n}}\| - \mathbb{E}\,\|S_{\mathbf{n}}\||^q \le C(q) \sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\,\|X_{\mathbf{k}}\|^q,$$

for $q = 2$ *constant* $C(2) = 4$;

(ii) if $q > 2$ *then*

$$\mathbb{E}\,|\|S_{\mathbf{n}}\| - \mathbb{E}\,\|S_{\mathbf{n}}\||^q \le C(q) \left[\left(\sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\,\|X_{\mathbf{k}}\|^2\right)^{q/2} + \sum_{\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\,\|X_{\mathbf{k}}\|^q\right]$$

are true.

Brunk-Prokhorov SLLN in separable Banach space

Application of Lemma 6.2.1 allows to generalize Acosta's result (cf. [1], Theorem 3.2) to the fields of random elements.

Theorem 6.2.2. ([86], Theorem 2.12) *Let* $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *be a random field of zero-mean random variables satisfying assumptions of Lemma 6.2.1, and* $\|S_{\mathbf{n}}\|/|\mathbf{n}| \stackrel{P}{\to} 0$ *as* $\max \mathbf{n} \to \infty$, *thus:*

(i) *if* $1 \le q \le 2$ *then* $\sum\limits_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbb{E}\,\|X_{\mathbf{n}}\|^q}{|\mathbf{n}|^q} < \infty$ *implies SLLN (77);*

(ii) *if* $q \ge 2$ *then* $\sum\limits_{\mathbf{n}\in\mathbb{N}^d} \frac{\mathbb{E}\,\|X_{\mathbf{n}}\|^q}{|\mathbf{n}|^{\frac{q}{2}+1}} < \infty$ *implies SLLN (77).*

Proof. (i). Let us assume that $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ are symetric (desymetryzation is standard) and put

$$Y_{\mathbf{k}} = X_{\mathbf{k}} I(\|X_{\mathbf{k}}\| \le |\mathbf{k}|), \quad T_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathbb{N}^d} Y_{\mathbf{k}}.$$

By assumption it follows that $\sum\limits_{\mathbf{n}\in\mathbb{N}^d} P(\|X_{\mathbf{k}}\| \le |\mathbf{k}|) < \infty$ and by the Borell-Cantelli Lemma it is enough to prove $T_{\mathbf{n}}/|\mathbf{n}| \to 0$ a.s. as $\max \mathbf{n} \to \infty$. It follows from assumption that

$$T_{\mathbf{n}}/|\mathbf{n}| \to 0 \quad in \quad \text{probability} \quad \text{as} \quad \max \mathbf{n} \to \infty,$$

thus by Lemma 4

$$E\|T_{\mathbf{n}}\|/|\mathbf{n}| \to 0 \quad \text{as} \quad \max \mathbf{n} \to \infty$$

and on the virtue of Lemma 3 and the Borell-Cantelli Lemma, the proof will be completed if we show that for any $\lambda > 0$

$$\sum_{\mathbf{n}\in\mathbb{N}^d} P\left(\frac{\|V_{\mathbf{k}}\|}{|2^{\mathbf{k-1}}|} > \lambda\right) < \infty \quad \text{where} \quad V_{\mathbf{k}} = \|T_{2^{\mathbf{k-1}}}^{2^{\mathbf{k}}}\| - E\|T_{2^{\mathbf{k-1}}}^{2^{\mathbf{k}}}\|.$$

Now, for any $\lambda > 0$ by Chebyshev inequality and Theorem 2.1 of Acosta [1] we have

$$\sum_{\mathbf{k}\in\mathbb{N}^d} P\left(\frac{\|V_{\mathbf{k}}\|}{|2^{\mathbf{k-1}}|} > \lambda\right) \le \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{E\|V_{\mathbf{k}}\|^p}{\lambda^p |2^{\mathbf{k-1}}|^p}$$
$$\le \frac{C_p 2^{rp}}{\lambda^p} \sum_{\mathbf{k}\in\mathbb{N}^d} \sum_{2^{\mathbf{k-1}} \preceq \mathbf{j} \prec 2^{\mathbf{k}}} \frac{E\|Y_{\mathbf{j}}\|^p}{|\mathbf{j}|^p} \le \frac{C_p 2^{rp}}{\lambda^p} \sum_{\in\mathbb{N}^d} \frac{E\|X_{\mathbf{k}}\|^p}{|\mathbf{k}|^p} < \infty.$$

Let us observe that we could use Theorem 2.1 of Acosta [1] in multidimensional indices case since $\{X_{\mathbf{k}},\ \mathbf{k} \in \mathbb{N}^d\}$ are independent.

(ii). The same arguments and Hölder inequality

$$\sum_{\mathbf{k}\in\mathbb{N}^d} P\left(\frac{\|V_{\mathbf{k}}\|}{|2^{\mathbf{k-1}}|} > \lambda\right) \le \frac{C_p}{\lambda^p} \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{1}{|2^{\mathbf{k-1}}|^p} \left[\left(\sum_{2^{\mathbf{k-1}} \preceq \mathbf{j} \prec 2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^2\right)^{\frac{p}{2}} + \sum_{2^{\mathbf{k-1}} \preceq \mathbf{j} \prec 2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^p\right]$$
$$\le \frac{C_p}{\lambda^p} \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{1}{|2^{\mathbf{k-1}}|^p} \left[|2^{\mathbf{k-1}}|^{\frac{p}{2}-1} \sum_{2^{\mathbf{k-1}} \preceq \mathbf{j} \prec 2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^p + \sum_{2^{\mathbf{k-1}} \preceq \mathbf{j} \prec 2^{\mathbf{k}}} E\|Y_{\mathbf{j}}\|^p\right]$$
$$\le \frac{2C_p}{\lambda^p} 2^{r(\frac{p}{2}+1)} \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{E\|X_{\mathbf{k}}\|^p}{|\mathbf{k}|^{\frac{p}{2}+1}} < \infty.$$

□

Chapter 7

Feller SLLN for fields of random elements

In this part of the outline we will present the generalization of Feller's strong law of large numbers for the fields of random elements taking values in Banach's space $(\mathbb{B}, \| \, \|)$, studying the limiting behavior of sums of random elements indexed by certain subsets of $\mathbb{N}^d$.

Let us start with additional notations and conditions. Let $\{a_{\mathbf{n}}, \in \mathbb{N}^d\}$ be such a field of positive numbers that $a_{\mathbf{n}} \to \infty$ as $\max \mathbf{n} \to \infty$, for which there exists an expanding sequence $\{D_k, \ k \ \in \mathbb{N}\}$, of finite subsets of $\mathbb{N}^d$ having the following properties:

(A) let $I_k := D_k - D_{k-1}, \ k \ \geq 1$, if $\mathbf{n} \in I_k$, then $(\mathbf{n}) \subseteq D_k$;

(B) there exist such constants $\tau > 1$ and $C_1, C_2 > 0$, that for any k and $\mathbf{n} \in I_k$, the condition $C_1 \tau^k \ \leq a_{\mathbf{n}} \leq C_2 \tau^{;}$ is met;

(C) for any k there exists a family of disjoint rectangles E_{kl} and a corresponding set of indices R_k such that $I_k = \bigcup\limits_{l \in R_k} E_{kl}$;

(D) $\nu_0 \limsup \max\limits_{\mathbf{n} \in I_k} \tau^{-k} \sum\limits_{i=1} \tau^k \left|\{t \in R_i : E_{it} \cap (\mathbf{n}) \neq 0\}\right| < \infty.$

Conditions (A)–(D) were introduced by Mikosh and Norvaiša in [96]. This property of the numeric field was called the *weak star property* (WSP). The condition seems strange and complicated, but reduced to one dimension "behaves as needed"; when $d = 1$, the increasing to infinity sequence $\{a_{\mathbf{n}},\ \in \mathbb{N}\}$ satisfies the WSP condition and assumption (85) implies a known Feller condition $\sum_{k\ \geq n} a_k^{-2} = O(n/a_n^2)$.

It turns out (cf. [95]) that in order to obtain the SLLN it is enough that the sum of random elements indexed with E_{kl} sets satisfies a certain asymptotic condition (this is also a necessary condition).

Lemma 7.0.1. ([76], Lemma 2.2) *If* $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ *is a field of independent, symmetric random elements with values in Banach's B space and the following two conditions are met:*

(i) $|X_{\mathbf{k}}| \leq a_{\mathbf{k}},\ \ \mathbf{k} \in \mathbb{N}^d$*;*

(ii) $\lim(\max) S_{\mathbf{n}}/a_{\mathbf{n}} \xrightarrow{\mathbb{P}} 0,$

then

$$\lim_{k\ \to\infty} \mathbb{E}\, \|S_{E_{kl}}/d^k\ \|^p \to 0,\ \ k\ \to \infty,\ \ \textit{uniformly in}\ \ l \in R_k, \tag{84}$$

for each $p > 0$.

Using the above lemma, we can obtain a generalization of Feller's SLLN (cf. [76], Theorem 3.1). In order to formulate this result, let us introduce the notations: $M_j := card\{\mathbf{n} \in \mathbb{N}^d :\ a_{\mathbf{n}} \leq j\}$ and $m_j := M_j - M_{j-1}$, for each $j \geq 1$.

Theorem 7.0.2. ([76], Theorem 3.1) *Suppose that there exists the natural number* j_0 *and such positive constants* $C_3,\ C_4$ *that for each* $j \geq j_0$

$$M_j \leq C_3 M_{j-1},\ \ \sum_{i\ \geq j} i^{-3} M_i\ \leq C_4 j^{-2} M_j. \tag{85}$$

If $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ *is a field of equidistributed random elements with values in Banach space* $(\mathbb{B}, \|\ \|)$, $\{a_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ *a non-negative real field increasing to infinity, then the following two conditions*

$$\lim(\max) S_{\mathbf{n}}/a_{\mathbf{n}} \xrightarrow{\mathbb{P}} 0, \tag{86}$$

$$\sum_{\mathbf{n}\in\mathbb{N}^d} \mathbb{P}(|X_{\mathbf{n}}| \geq a_{\mathbf{n}}) < \infty, \tag{87}$$

are equivalent to the one below

$$\lim(\max) S_{\mathbf{n}}/a_{\mathbf{n}} \to 0 \quad a.s. \tag{88}$$

The above result allows for many applications. Setting $a_{\mathbf{n}} = |\mathbf{n}|^{1/p},\ 1 \leq p \leq 2$, and assuming (85) we get Marcinkiewicz SLLN for fields of random elements, proved by Fazekas in [40].

Corollary 7.0.3. *If we assume that* $a_{\mathbf{n}} = n_1^{1/p_1} \cdot \dots \cdot n_d^{1/p_d}$, *where* $\mathbf{1} \preceq \mathbf{p} \prec \mathbf{2}$, *we can obtain results corresponding to the results presented in chapter 3; that is, we get the necessary and sufficient conditions for the strong law of large numbers with asymmetric normalization.*

Note also that the claim remains its truth when, in the assumptions, the weak law of large numbers is replaced by the conditions regarding the geometry of Banach space (see [76], Theorem 3.2).

Chapter 8

Weak convergence of random fields

This chapter is based on the results obtained in [75], in which we deal with the weak convergence of fields of random elements with values in the metric space. Such problems arise when examining the weak convergence of stochastic processes, empirical processes or randomly stopped empirical sums created from samples from a continuous distribution with "time" in $\mathbb{R}^q$. In a more general context, such results can be the basis for the application of random fields in biology, the propagation of electromagnetic waves generated in a medium of random parameters or in the study of turbulent flows.

Theorems from [75] generalize or complement the results contained in the publications: [3], [4], [9], [10], [29] and [31].

8.1. Weak convergence of random fields with random indices

We will consider the random field $\{Y_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ defined on the probabilistic space $(\Omega, \mathfrak{F}, \mathbb{P})$, with values in the separable metric space (S, ρ), with the σ-algebra of the

Borel sets $\mathfrak{B}$. Let us also add $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$ will be a d-dimensional field of random variables defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, more precisely $\mathbf{N_n} = (N_{\mathbf{n}}^{(1)}, N_{\mathbf{n}}^{(2)}, \ldots, N_{\mathbf{n}}^{(d)})$, where $N_{\mathbf{n}}^{(i)}$ for each $1 \leq i \leq d$ and $\mathbf{n} \in \mathbb{N}^d$ is a random variable assuming natural values.

Let Y be a random element with values in the metric space (S, ρ) with a distribution of μ and

$$\lim(\min) Y_{\mathbf{n}} \xrightarrow{\mathfrak{D}} \mu. \tag{89}$$

We give the sufficient conditions (in most cases also necessary) that the random field $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ should meet to ensure the convergence

$$\lim(\min) Y_{N_{\mathbf{n}}} \xrightarrow{\mathfrak{D}} \mu, \tag{90}$$

without imposing any conditions on the probabilistic relations between random fields $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ and field of random index $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$.

Let $\{k_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a collection of positive numbers such that $k_{\mathbf{n}} \to \infty$ as $\max \mathbf{n} \to \infty$, furthermore we assume that $\{k_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is non-decreasing in the sense that for every $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$, $k_{\mathbf{m}} \leq k_{\mathbf{n}}$ provided $\mathbf{m} \preceq \mathbf{n}$. Now, let us introduce the following definition witch generalized Anscombe condition. Note, that in the $d = 1$ the concept of norming sequence and generalized Anscombe condition was introduced in [29] by Csörgő and Rychlik.

Definition 8.1.1. ([75], Definition 1) *A random field $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to satisfy the following generalized Anscombe condition with norming family $\{k_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ of positive real numbers such that for every $\varepsilon > 0$ there exists $\delta > 0$*

$$\limsup_{\min \mathbf{n} \to \infty} \mathbb{P}\Big(\max_{\preceq \mathbf{n} \in D_{\mathbf{n}}(\delta)} \rho(Y_{\preceq \mathbf{n}}, Y_{\mathbf{n}}) \geq \varepsilon\Big) \leq \varepsilon, \tag{91}$$

where

$$D_{\mathbf{n}}(\delta) := \{\preceq \mathbf{n} \in \mathbb{N}^d : |k_{\preceq \mathbf{n}} - k_{\mathbf{n}}| \leq \delta k_{\mathbf{n}}\}.$$

Now, we are ready to state extension of Aldous'theorem to random field.

Theorem 8.1.2. ([75], Theorem 1) *Let* $\{Y_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *be the random field, then the following conditions are equivalent: If we assume that the random field* $\{Y_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *satisfies the following generalized Anscombe condition:*

(i) $\{Y_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *satisfies generalized Anscombe condition with* $\{k_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *and* $\lim(\min) Y_{N_{\mathbf{n}}} \xrightarrow{\mathfrak{D}} \mu$;

(ii) $\lim(\min) Y_{N_{\mathbf{n}}} \xrightarrow{\mathfrak{D}} \mu$ *for every random field* $\{N_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ *such that*

$$\lim(\min) k_{\mathbf{N_n}} / k_{\mathbf{a_n}} \xrightarrow{\mathbb{P}} 1, \tag{92}$$

for a some field $\{\mathbf{a_n},\, \mathbf{n} \in \mathbb{N}^d\}$ *taking values in* $\mathbb{N}^d$ *and such that* $\min \mathbf{a_n} \to \infty$, *as* $\min \mathbf{n} \to \infty$.

Interpretation of the field $\{k_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$, which we used in the generalized Anscombe condition, one can find in the from Theorem 1, given in [75]. Namely, if we set that $Y_{\mathbf{n}} := S_{\mathbf{n}}/B_{\mathbf{n}}$, where $S_{\mathbf{n}} = \sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}$ and $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is the field of independent random variables, such that $\mathbb{E}\, X_{\mathbf{n}} = 0$, $\mathbb{E}\, S_{\mathbf{n}}^2 = B_{\mathbf{n}}^2 < \infty$, and $B_{\mathbf{n}} \to \infty$ as $\min \mathbf{n} \to \infty$; then (89) and (90) are equivalent when in the Anscombe condition we use the field $k_{\mathbf{n}} = B_{\mathbf{n}}^2$.

In this part of the subsection we will present sufficient and necessary conditions for convergence (90), when we weaken the condition (92), imposed on the random field $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$, at the expense of assumptions on the random field $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$, about which we will assume that

$$\lim(\min) Y_{\mathbf{n}} \xrightarrow{\mathfrak{D}} \mu \ \text{(stably)}. \tag{93}$$

This time a condition analogous to the Anscombe condition holds for any event $A \in \mathfrak{B}$:

$$\limsup_{\min \mathbf{n} \to \infty} \mathbb{P}_A\big(\max_{\preceq \mathbf{n} \in D_{\mathbf{n}}(\delta)} d(Y_{\preceq \mathbf{n}}, Y_{\mathbf{n}}) \geq \varepsilon \big) \leq \varepsilon \mathbb{P}(A), \tag{94}$$

whereas the random field $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$ must satisfy the condition that requests that for any $\varepsilon > 0$ and $\delta > 0$ there be a finite and measurable partition of the space Ω into disjoint events $\{A_1, \ldots, A_M\}$ and a d-dimensional field $\{\mathbf{a}_{\mathbf{n}_j} \in \mathbb{N}^d\}$, $1 \le j \le M$, $\mathbf{n} \in \mathbb{N}^d$, such that $\min \mathbf{a}_{\mathbf{n}_j} \to \infty$ as $\min \mathbf{n}_j \to \infty$, and the following inequality holds

$$\limsup_{\min \mathbf{n} \to \infty} \sum_{j=1}^{M} \mathbb{P}_{A_j}(|k_{\mathbf{N_n}} - k_{\mathbf{a}_{\mathbf{n}_j}}| > \delta k_{\mathbf{a}_{\mathbf{n}_j}}) \le \varepsilon, \tag{95}$$

where $\{k_{\mathbf{n}},\, \mathbf{n} \in \mathbb{N}^d\}$ is the field as in the Anscombe condition (91). In [75] we point out that if we assume (93) and maintain the other assumptions, we also get a stable convergence in (90). On the other hand, as Theorem 3 from [75] shows, in order to obtain (90) with condition (95), we cannot weaken the assumption $Y_{\mathbf{n}} \xrightarrow{\mathfrak{D}} \mu$ (stably). The results shown, mainly generalized the Aldous theorems contained in [3] to non-stationary random fields.

8.2. Random Functional Central Limit Theorem

In this part of the chapter we present a functional central limit theorem, for random elements taking values in a certain metric space and having a multidimensional random indices. If this metric space is a set of all functions defined on $T_d = \langle 0, \infty)^d$, "continuous from above and having limits from below" (cf. def. in [79]), then random elements with values in this space are a very important class of stochastic processes. The results for the low convergence of such random elements may be used in the analysis of the weak convergence of empirical processes.

The results we are discussing generalize the main theorems in [4], [9], [10], [29] and [31]; to introduce them more accurately, we need some additional notations. On the set of all the functions "continuous from above and having limits from below" defined on the set T_d, we introduce the metric (see def. in [?]); the resulting metric space is denoted $(D_d\,[0,\infty)\,, \varrho)$, it turns out to be separable and complete (cf. [79]). In the further part of this subsection, the product of vectors and d-dimensional fields will be understood as a matrix multiplication, i.e.; let $\mathbf{q}, \mathbf{t} \in T_d$ and $0 < |\mathbf{q}| < \infty$ and $\mathbf{p_n} = (p_{\mathbf{n}}^{(1)}, ..., p_{\mathbf{n}}^{(d)})$, where $p_{\mathbf{n}}^{i} \in T_d$ for $1 \le i \le d$, then

$$\mathbf{q} \cdot \mathbf{t} := (q_1 t_1, ..., q_d t_d), \;\; \mathbf{t} \cdot \mathbf{p_n} := (t_1 p_{\mathbf{n}}^{(1)}, ..., t_d p_{\mathbf{n}}^{(d)}).$$

For an arbitrary $0 < \alpha < \infty$ we define the mapping

$$F_{\mathbf{q}}x(t) = |\mathbf{q}|^{-\alpha}x(q_1t_1, ..., q_dt_d),$$

with domain and co-domain equal $D_d\,[0, \infty)$.

Let $Z = Z(\mathbf{t})$, $\mathbf{t} \in T_d$, be a random element taking values in $D_d\,[0, \infty)$, $\{\mathbf{k_n} = (k_{\mathbf{n}}^{(1)}, ..., k_{\mathbf{n}}^{(d)}),\ \mathbf{n} \in \mathbb{N}^d\}$ such d-dimensional, positive real field, that $|\mathbf{k_n}| = \prod_{i=1}^{d} k_{\mathbf{n}}^{(i)} \to \infty$ as $\min \mathbf{n} \to \infty$ and $\mathbf{n} \preceq \mathbf{m}$ implies $|\mathbf{k_n}| \leqq |\mathbf{k_m}|$.

Theorem 8.2.1. ([75], Theorem 3) *Let for an arbitrary* α, $0 < \alpha < \infty$, *every* $\mathbf{t} \in T_d$ *and* $\mathbf{k_n} \in T_d^+$ *the field of random elements be defined as follows*

$$Y_{\mathbf{n}}(\mathbf{t}) := |\mathbf{k_n}|^{-\alpha}Z(\mathbf{t} \cdot \mathbf{k_n}), \quad \mathbf{n} \in \mathbb{N}^d. \tag{96}$$

Then, or such random field, the following conditions are equivalent :

(i) $\lim_{\min \mathbf{n} \to \infty} Y_{\mathbf{n}} \xrightarrow{\mathfrak{D}} \mu$ *(stably),*

(ii) $\lim_{\min \mathbf{n} \to \infty} Y_{\mathbf{N_n}} \xrightarrow{\mathfrak{D}} \mu$, *for any random field* $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$ *satisfying condition*

Let us quote a few remarks to this result, given in [75]:

- the processes considered in [9], [10] and [29] are special cases of the process defined by (96); [10] i [29]

- condition (95) is the weakest possible, at which we do not impose any conditions on the structure of dependence between random fields $\{Y_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ and $\{\mathbf{N_n},\ \mathbf{n} \in \mathbb{N}^d\}$;

- in the light of the above remark, the result obtained is the best possible in the aspect considered;

- the result generalized studies [9], [10] and [31];

- equivalence remains true if the considerations are limited to the $D_d[0, 1]$ space.

As an application, the following random functional limit theorem was obtained (cf. [75], Theorem 5]).

Theorem 8.2.2. *Let* $Y_{\mathbf{n}}(\mathbf{t}) := (|\mathbf{n}|)^{-1/2} \sum_{\mathbf{k} \preceq \mathbf{n} \cdot \mathbf{t}} X_{\mathbf{k}}$, *for* $\mathbf{t} \in T_d\,[0,1]$. *If* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *is such a stationary, ergodic field of martingale differences with respect to the filtration* $\{\mathfrak{F}_{\mathbf{n}} = \bigvee_{\mathbf{k} \preceq \mathbf{n}} \sigma(X_{\mathbf{k}}),\ \mathbf{n} \in \mathbb{N}^d\}$, *that* $\mathbb{E}\, X_{\mathbf{n}}^2 = 1$ *then*

$$\lim(\min) Y_{\mathbf{N_n}} \xrightarrow{\mathfrak{D}} W \quad \textit{in} \;\; D_d\,[0,1]\,,$$

for any field $\{\mathbf{N_n}, \mathbf{n} \in \mathbb{N}^d\}$ *satisfying condition* (95) *with the normalizing field* $\{k_{\mathbf{n}} = |n|,\ \mathbf{n} \in \mathbb{N}^d\}$, *where* W *is* d*-parameter Wiener process.*

Chapter 9

Rate of convergence in the random WLLN

According to the title of this subsection, we will provide the results giving the rate of convergence in the weak law of large numbers for fields of independent random variables and martingales, for their partial sums indexed randomly. More specifically, we determine the order of magnitude of

$$h(t) \sum_{\mathbf{n} \in \mathbb{N}^d} f(\mathbf{n}) \mathbb{P}\left(|S_{\mathbf{N_n}}| \geq t |\mathbf{N}_n|^{1/2} g(\mathbf{N_n}) \right), \text{ as } t \to 0^+, \tag{97}$$

where $\{\mathbf{N_n},\, \mathbf{n} \in \mathbb{N}^d\}$ is a random field with values in $\mathbb{N}^d$.

9.1. Results for fields of independent random variables

Due to the broadness of the formulation of the results that we want to present, we will limit their discussion to the general idea, omitting the details.

Gut in [51] proved a random version of the Baum-Katz theorem. With the assumptions similar to the classic version of this result – the condition

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} \mathbb{P}(|N_n - \lambda n| > tn) < \infty, \tag{98}$$

where λ jest is such a random variable that $\mathbb{P}(a \le \lambda \le b) = 1$ for some $0 < a < b \le \infty$, implies

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\mathbb{P}(|S_{N_n}| > tN_n^{\alpha}) < \infty. \tag{99}$$

This result was the premise of our research, and was generalized not only towards random fields and dependence structure, but we also give an order of magnitude of a more general sum. Let us denote:

$$H(r,s;t) = \sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r (log_+|\mathbf{n}|)^s \mathbb{P}(|\mathbf{N_n}| - \lambda|\mathbf{n}|| > t|\mathbf{n}|), \tag{100}$$

where λ is such a random variable as in condition (98).

Theorem 9.1.1. *If*

$$\lim(\max)\sup_{x\in\mathbb{R}} |\mathbb{P}(S_{\mathbf{N_n}} < x|\mathbf{N_n}|^{1/2}) - \Phi(x)| \to 0, \tag{101}$$

then with the assumptions analogous to Gut's theorem from [51], there exists such a constant C depending on the function: f, g, h and the dimension d of the index set that

$$\liminf_{t\to 0^+} h(t)\Big\{ \sum_{\mathbf{n}\in\mathbb{N}^d} f(\mathbf{n})\mathbb{P}\left(|S_{\mathbf{N_n}}| \ge t|\mathbf{N}_n|^{1/2} g(\mathbf{N_n})\right) + H(r,s;t)\Big\} \ge C_{f,g,h;d}$$

and

$$\limsup_{t\to 0^+} h(t)\Big\{ \sum_{\mathbf{n}\in\mathbb{N}^d} f(\mathbf{n})\mathbb{P}\left(|S_{\mathbf{N_n}}| \ge t|\mathbf{N}_n|^{1/2} g(\mathbf{N_n})\right) - H(r,s;t)\Big\} \le C_{f,g,h;d}.$$

The exact specification of functions and assumptions is contained in Theorem 2, given in [74].

9.2. Results for martingale random fields

As in the previous subsection, we will examine the asymptotics of (97); here we consider the random field $\{X_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$ of martingale differences. For the same reasons as before, we will only give the most important facts about the results. However, we will not avoid some new notations:

$$F_a(\alpha,r,s,u) := \sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r (log^+|\mathbf{n}|)^s \Phi\left(-ta_{\mathbf{n}}^{\alpha}(t)(\log^+ a_{\mathbf{n}}(t))^u\right),$$

$$F_b(\alpha,r,s,u) := \sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r (log^+|\mathbf{n}|)^s \Phi\left(-tb_{\mathbf{n}}^{\alpha}(t)(\log^+ b_{\mathbf{n}}(t))^u\right),$$

where Φ is a normal distribution,

$$a_{\mathbf{n}} := (a-t)\zeta_{\mathbf{n}}^2, \qquad b_{\mathbf{n}} := (b+t)\zeta_{\mathbf{n}}^2,$$
$$\zeta_{\mathbf{n}}^2 := \sum_{\mathbf{k}\preceq\mathbf{n}} \sigma_{\mathbf{k}}^2 \quad\text{and}\quad \sigma_{\mathbf{k}}^2 := \mathbb{E}\,X_{\mathbf{k}}^2.$$

Theorem 9.2.1. *If the considered random field of martingale differences satisfied (101) and there exist such positive number fields*

$$\{b_{y_{\mathbf{k}}}^2,\ \mathbf{k}\in\mathbb{N}^2\} \text{ and } \{a_{y_{\mathbf{k}}}^r,\ \mathbf{k}\in\mathbb{N}^2\},$$

that for $\mathbf{j}\preceq\mathbf{k}\preceq\mathbf{n}$ *we have*

$$\mathbb{E}\,(X_{\mathbf{k}}^2\mathbb{I}(X_{\mathbf{k}}\le y_{\mathbf{k}})|\mathfrak{F}_{\mathbf{j}}) \le b_{y_{\mathbf{k}}}^2 \text{ a.s.}$$

and

$$\mathbb{E}\,(X_{\mathbf{k}}^r\mathbb{I}(0\le X_{\mathbf{k}}\le y_{\mathbf{k}})|\mathfrak{F}_{\mathbf{j}}) \le a_{y_{\mathbf{k}}}^r \text{ a.s.},$$

thus, with these assumptions, if

$$\max[t, \ln(\beta x y^{t-1}/A_{t,Y}+1)] > \alpha x y/(e^t B_Y^2),$$

then

$$\liminf_{t\to 0^+} h(t)\left(\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r (log_+|\mathbf{n}|)^s \mathbb{P}(|S_{N_{\mathbf{n}}}| \ge tM_{\mathbf{n}}g(M_{\mathbf{n}})) + \widetilde{H}(r,s;t)\right)$$
$$\ge \liminf_{t\to 0^+} F_b(\alpha,r,s,u)h(t)$$

and

$$\limsup_{t\to 0^+} h(t)\left(\sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r(log_+|\mathbf{n}|)^s\mathbb{P}(|S_{N_\mathbf{n}}| \geq tM_\mathbf{n}g(M_\mathbf{n})) - \widetilde{H}(r,s;t)\right) \\ \leq \limsup_{t\to 0^+} F_a(\alpha,r,s,u)h(t)$$

where

$M_\mathbf{n} = \sum_{\mathbf{k}\preceq\mathbf{N_n}} \sigma_\mathbf{k}^2$, $\widetilde{H}(r,s;t) = \sum_{\mathbf{n}\in\mathbb{N}^d} |\mathbf{n}|^r(log_+|\mathbf{n}|)^s\mathbb{P}(|M_\mathbf{n} - \lambda\zeta_\mathbf{n}| > t\zeta_\mathbf{n}^2)$ *and* λ *is such a random variable as in condition (98).*

The complete formulation of the results are given in [77].

Index of Symbols

$:=$ – equal by definition;

$(\Omega, \mathfrak{F}, \mathbb{P})$ – probability space, Ω– set, $\mathfrak{F}$ – σ-algebra of subsets of Ω,
$\mathbb{P}$ – probability;

$\mathbb{N}$ – set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

$\mathbb{N}^d := \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ – product of d sets X;

$\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$, $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$;

$D = \{1, 2 \dots, d\}$, $\emptyset \neq J \subseteq D$ i $CJ := D \setminus J$;

$\mathbf{m} \preceq \mathbf{n} \iff m_i \leq n_i$ for every $i \in D$;

$\mathbf{m} \wedge \mathbf{n} := (m_1 \wedge n_1, \dots, m_d \wedge n_d)$;

X, $X(\omega)$ – random variables on $(\Omega, \mathfrak{F}, \mathbb{P})$;

$\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ – random fields;

$S_{\mathbf{n}} := \sum_{\mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}}$

$\min \mathbf{n} := \min\limits_{i \in D} n_i$;

$\max \mathbf{n} := \max\limits_{i \in D} n_i$;

$|\mathbf{n}| := \prod\limits_{i \in D} n_i$;

$\|\mathbf{n}\|_D := \max\limits_{i \in D} |n_i|$;

$(\mathbf{n}) := \{\mathbf{k} \in \mathbb{N}^d :\ \mathbf{k} \preceq \mathbf{n}\}$;

$\mathfrak{F}_{\mathbf{n}}^J := \bigvee\limits_{(n_j \in \mathbb{N}, j \in CJ)} \mathfrak{F}_{\mathbf{n}},\ \mathfrak{F}_{\mathbf{n}}^J := \sigma\{\bigcup\limits_{(n_j \in \mathbb{N}, j \in CJ)} \mathfrak{F}_{\mathbf{n}}\}$;

$\mathfrak{F}_{\mathbf{n}}^j = \mathfrak{F}_{\mathbf{n}}^J$ if $J = \{j\}$;

$\mathcal{G}_{\mathbf{n}} := \bigvee\limits_{j=1}^{d} \mathfrak{F}_{\mathbf{n}}^j$;

$\widetilde{\mathfrak{F}}_{\mathbf{n-1}} := \mathcal{G}_{\mathbf{n-1}} \wedge \mathfrak{F}_{\mathbf{n}}$, where $\mathbf{n-1} := (n_1-1, n_2-1, ..., n_d-1)$;

$\boldsymbol{\alpha} := (\alpha_1, \alpha_2, ..., \alpha_d)$;

$\mathbf{n}^{\boldsymbol{\alpha}} := (n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_d^{\alpha_d}), \;\; |\mathbf{n}^{\boldsymbol{\alpha}}| := n_1^{\alpha_1} \cdot n_2^{\alpha_2} \cdot \ldots \cdot n_d^{\alpha_d}$;

$p := \max\{k : \alpha_k = \alpha_1\}$; $\mathcal{S}_\theta^d = \left\{(i_1, ..., i_d) \in \mathbb{N}^d : \theta < \frac{i_l}{i_k} < \frac{1}{\theta}, \text{ for all } l \neq k \in D\right\}$;

$d_\theta(k) = \text{card}\left\{\mathbf{n} \in \mathcal{S}_\theta^d : |\mathbf{n}| = k\right\}, \; k \in \mathbb{N}$;

$M_\theta(k) = \text{card}\left\{\mathbf{n} \in \mathcal{S}_\theta^d : |\mathbf{n}| \leq k\right\}, \; k \in \mathbb{N}$;

$\widehat{\mathcal{S}}_\theta^d\left(\mathbf{n}\right) := \mathcal{S}_\theta^d \backslash \left(\mathbf{n}\right), \;\; \mathbf{n} \in \mathcal{S}_\theta^d$;

$\mathcal{S}_\theta^d(\mathbf{n}) := \mathcal{S}_\theta^d \cap \left(\mathbf{n}\right), \;\; \mathbf{n} \in \mathcal{S}_\theta^d$;

$\mathcal{S}_\theta^d(|\mathbf{n}|) := \left\{\mathbf{i} \in \mathcal{S}_\theta^d : |\mathbf{i}| \leq |\mathbf{n}|\right\}, \;\; \mathbf{n} \in \mathcal{S}_\theta^d$;

$Y_{\mathbf{n}} \xrightarrow{\mathfrak{D}} \mu$ weak convergence as $\min$ or $\max \mathbf{n} \to \infty$;

$\mathbf{i} \vee \mathbf{k} := (i_1 \vee k_1, \ldots, i_d \vee k_d)$;

$\mathcal{G}'_{\mathbf{k+1}} := \bigvee_{\mathbf{j} \not\geq \mathbf{k}} \mathfrak{F}'_{\mathbf{j}}$;

Bibliography

[1] A. de Acosta, Inequalities for B-valued random vectors with applications to the strong law of large numbers, *Ann. Probab.*, 9 (1981), 157–161.

[2] O. Ahmad, J.-C. Pinoli, On the linear combination of the Gaussian and student's t random field and the integral geometry of its excursion sets, *Statist. Probab. Lett.*, 83 (2013), 559–567.

[3] D.J. Aldous, Weak convergence of randomly indexed sequences of random variables, *Math. Proc. Camb. Phil. Soc.*, 83 (1978), 117–126.

[4] F.J. Anscombe, Large-sample theory of sequential estimation, *Proc. Camb. Philos. Soc.*, 48 (1952), 600–607.

[5] H.A. Azarnoosch, A. Bozorgnia, The strong law of large numbers for pairwise negatively dependent random variables, *Iran. J. Sci. Technol. Trans. A Sci.*, 28 (2004), 211–217.

[6] K.A. Astbury, The order convergence of martingales indexed by directed sets, *Trans. Amer. Math. Soc.* 265 (1981),495–510.

[7] R. Bass, R. Pyke, A strong law of large numbers for partial sum processes indexed by sets, *Ann. Probab.* 12 (1984), 268–271.

[8] R. Bass, R. Pyke, Functional law of the iterated logarithm and uniform central limit theorem for partial-sum processes indexed by sets, *Ann. Probab.*, 12 (1984), 13–34.

[9] A.K. Basu, C.C.Y. Dorea, On functional central limit theorem for stationary martingale random fields, *Acta Math. Hung.*, 33(1979), 307–316.

[10] P.J. Bickel, M.J. Wichura, Convergence criteria for multiparameter stochastic processes and some applications, *Ann. Math. Stat.*, 42 (1971), 1656–1670.

[11] S. Bochner, Partial ordering in the theory of martingales, *Ann. of Math.*, 62 (1955), 162–169.

[12] V.M. Borodikhin, On generalization of the Nagaev-Fuk inequalities for a class of random fields, *Sib. Math. J.*, 36 (1995), 1101–1107.

[13] A. Bozorgnia, R.F. Patterson, R.L. Taylor, Limit theorems for ND random variables, *Technical Report*, University of Georgia (1993).

[14] A. Bulinski, A. Shashkin, Limit theorems for associated random fields and related systems, *World Scientific*, Hackensack, NJ (2007).

[15] D.L. Burkholder, A maximal inequalities as necessary condition, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 3 (1964), 75–88.

[16] R. Cairoli, Une inégalité pour martingales à indices multiples et ses applications, *Lecture Notes in Mathematics*, Springer, Berlin, 124 (1970), 1–27.

[17] R. Cairoli, J.B. Walsh, *Acta Math.* Stochastic integrals in the plane, 134 (1975), 111–183.

[18] J. Cao, K.J. Worsley, Applications of random fields in human brain mapping, *Lecture Notes in Statistics*, Springer 159 (2001), 170–182.

[19] J. Cao, K.J. Worsley, The geometry of correlation fields with an application to functional connectivity of the brain, *Ann. Appl. Probab.*, 9 (1999), 1021–1057.

[20] F. Carbonell, K.J. Worsleya, N.J. Trujillo-Barreto, M. Vega-Hernandez, The geometry of time-varying cross-correlation random fields, *Comput. Stat. Data Anal.*, 53 (2009), 3291–3304.

[21] S.D. Chatterji, Comments on the martingale convergence theorem, *Symposium on Probability Methods in Analysis*, Loutraki (1967), 55–61.

[22] S.D. Chatterji, Martingale convergence and the Radon-Nikodym theorem in Banach spaces, *Math. Scand.*, 22 (1968), 21–41.

[23] T.C. Christofides, R.J. Serfling, Maximal inequalities for multidimensionally indexed submartingale arrays, *Ann. Probab.* 18 (1990), 630–641.

[24] Y. Chen, C.Y. Kam, Sums of pairwise quasi-asymptotically independent random variables with consistent variation, *Stochastic Models*, 25 (2009), 76–89.

[25] D. Cheng, Randomly weighted sums of dependent random variables with dominated variation, *J. Math. Anal. Appl.*, 420 (2014), 1617–1633.

[26] K.L. Chung, A course in probability theory, 2nd ed., *Academic Press*, New York-London (1974).

[27] J.A. Cuesta, C. Matran, On the asymptotic behavior of sums of pairwise independent random variabies, *Statist. Probab. Lett.*, 11 (1991), 201–210.

[28] S. Csörgő, K. Tandori, V. Totik, On the strong law of large numbers for pairwise independent random variables, *Acta Math. Hung.* 42 (1983), 319–330.

[29] M. Csörgő, Z. Rychlik, Weak convergence of sequences of random elements with random indices, *Math. Proc. Cambridge Philos. Soc.* 88 (1980) , 171–174.

[30] S. Csörgő, K. Tandori, V. Totik, On the convergence of series of pairwise independent random variables, *Acta Math. Hung.*, 45 (1985), 445–450.

[31] C.M. Deo, A Functional Central Limit Theorem for Stationary Random Fields, *Ann. Probab.*, 3 (1975), 708–715.

[32] J. Dieudonné, Sur un thèoremé de Jessen, *Fund. Math.*, 37 (1950), 242–248.

[33] Q. Dehua, K.C. Chang, R. Giuliano Antonini, A.Volodin, On the strong rates of convergence for arrays of rowwise negatively dependent random variables, *Stochastic Anal. Appl.*, 29 (2011), 375–385.

[34] L.E. Dubins, J. Pitman, A divergent, two-parameter, bounded martingale, *Proc. Am. Math. Soc.*, 78 (1980), 414–416.

[35] N. Dunford, An individual ergodic theorem for non-commutative transformations, *Acta Sci. Math.* (Szeged), 14 (1951), 1–4.

[36] G.A. Edgar, L. Sucheston, Stopping times and directed processes, *Cambridge Univ. Press* (1992).

[37] V.A. Egorov, A generalization of the Hartman-Wintner theorem on the law of the iterated logarithm, (Russian) *Vestn. Leningr. Univ., Math.* 7 (1971), 22–28.

[38] N. Etemadi, An elementary proof of the strong law of large numbers, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 55 (1981), 119–122.

[39] V. Fakoor, H.A. Azarnoosh, Probability inequalities for sums of negatively dependent random variables, *Pakistan J. Stat.*, 12 (2005), 257–264.

[40] I. Fazekas, Marcinkiewicz strong law of large numbers for $\mathbb{B}$-valued random variables with multidimensional indices, *Proc. of 3rd Pannonian Symp. on Math. Stat.* (1983), 53–61.

[41] I. Fazekas, O.I. Klesov, A general approach to the strong laws of large numbers, *Theory Probab. Appl.*, 45 (2002), 436–449.

[42] J.P. Gabriel, Loi des grands nombres, séries et martingales á deux indices, *C. R. Acad. Sci. Sér. A*, 279 (1974), 169–171.

[43] A. Gadidov, Sectorial convergence of U-statistics, *Ann. Probab.*, 23 (2005), 816–822.

[44] Q. Gao, N. Jin, H. Shen, Asymptotic behavior of the finite-time ruin probability with pairwise quasi-asymptotically independent claims and constant interest force, *Rocky Mt. J. Math.*, 44 (2014), 1505–1528.

[45] Q. Gao, N. Jin, Randomly Weighted Sums of pairwise quasi upper-tail independent increments with application to risk theory, *Commun. Stat., Theory Methods* 44 (2015), 3885–3902.

[46] Q. Gao, Y. Yang, Uniform asymptotics for the finite-time ruin probability in a general risk model with pairwise quasi-asymptotically independent claims and constant interest force, *Bull. Korean Math. Soc.*, 50 (2013), 611–626.

[47] S. Ghosal, T.K. Chandra, Complete convergence of martingale arrays, *J. Theor. Probab.* 11 (1998), 621–631.

[48] A. Gut, Marcinkiewicz laws and convergence rates in the law of large numbers with multidimensional indices, *Ann. Probab.*, 6 (1978), 469–482.

[49] A. Gut, Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices, *Ann. Probab.*, 8 (1980), 298–313.

[50] A. Gut, Strong laws for independent identically distributed random variables indexed by a sector, *Ann. Probab.*, 11 (1983), 569–577.

[51] A. Gut, Complete convergence and convergence rates for randomly indexed partial sums with an application to some first passage times, *Acta Math. Hung.*, 42 (1983), 225–232.

[52] A. Gut, U. Stadtmüller, Cesàro summation for random fields, *J. Theor. Probab.*, 23 (2010), 715–728.

[53] A. Gut, U. Stadtmüller, On the Hsu-Robbins-Erdös-Spitzer-Baum-Katz theorem for random fields, *J. Math. Anal. Appl.*, 387 (2012), 447–463.

[54] A. Gut, U. Stadtmüller, An asymmetric Marcinkiewicz–Zygmund LLN for random fields, *Statist. Probab. Lett.*, 79 (2009), 1016–1020.

[55] A. Gut, U. Stadtmüller, An intermediate Baum–Katz theorem, *Statistics & Probability Letters*, 81 (2011), 1486–1492.

[56] A. Gut, Probability, A Graduate Course, 2nd ed., Springer-Verlag, New York (2007).

[57] J. Hoffmann-Jørgensen, K.-L. Su, R.L. Taylor, The law of large numbers and the Ito-Nisio theorem for vector valued random fields, *J. Theor. Probab.*, 10 (1997), 145–183.

[58] T.-C. Hu, A. Rosalsky, A. Volodin, A complete convergence theorem for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces, *Stochastic Anal. Appl.*, 30 (2012), 343–353.

[59] R. Jajte, On the strong law of large numbers, *Ann. Probab.*, 31 (2003), 409–412.

[60] K.-H. Indlekofer, O.I. Klesov, Strong law of large numbers for multiple sums whose indices belong to a sector with function boundaries, *Theory Probab. Appl.*, 52 (2008), 711–719.

[61] K. Joag-Dev, F. Proschan, Negative association of random variables with applications, *Ann. Statist.*, 11 (1983), 286–295.

[62] D. Khoshnevisan, Multiparameter processes. An introduction to random fields, *Springer Monographs in Mathematics* xix (2002).

[63] O.I. Klesov, The Hájek-Rényi inequality for random fields and the strong law of large numbers, (Russian) *Teor. Veroyatnost. i Mat. Statist.*, 22 (1980), 58–66.

[64] O.I. Klesov, The three-series theorem for random fields with independent values, (Ukrainian) *Visnik Kiïv. Univ. Ser. Mat. Mekh.*, 22 (1980), 35–40.

[65] O.I. Klesov, The law of the iterated logarithm for multiple sums, (Russian) *Teor. Veroyatnost. i Mat. Statist.*, 27 (1982), 60–67.

[66] O.I. Klesov, The strong law of large numbers for multiple sums of independent identically distributed random variables, *Math. Notes*, 38 (1986), 1006–1014.

[67] O. Klesov, Z. Rychlik, The strong law of large numbers on partially ordered sets, *Theory Probab. Math. Stat.*, 58 (1999), 35–41.

[68] O.I. Klesov, Limit theorem for multi-indexed sums of random variables, Springer-Verlag (2014).

[69] M.-H. Ko, T.-S. Kim, H.-C. Kim, Strong laws of large numbers for asymptotically quadrant independent random variables, *Commun. Korean Math. Soc.*, 19 (2004), 765–773.

[70] K. Krickeberg, Convergence of martingales with a directed index set, *Trans. Amer. Math. Soc.*, 83 (1956), 313–337.

[71] K. Krickeberg, My encounters with martingales, *J. Contemp. Math. Anal., Armen. Acad. Sci.*, 44 (2009), 9 –13.

[72] V. Kruglov, A. Volodin, T.-C. Hu, On complete convergence for arrays, *Statist. Probab. Lett.*, 76 (2006), 1631–1640.

[73] A. Kuczmaszewska, On complete covergence in Marcinkiewicz-Zygmund type SLLN for negatively associated random variables, *Acta Math. Hung.*, 128 (2010), 116–130.

[74] Z.A. Łagodowski, Probabilities of moderate deviations for randomly indexed sums of random variables with multidimensional indices. *Probab. Math. Stat.*, 9, (1988), p. 115–123.

[75] Z.A. Łagodowski,, Z. Rychlik, Weak convergence of sequences of random elements with multidimensional random indices. *Publ. Math.*, 37 No.1-2 (1990), 35–40.

[76] Z.A. Łagodowski, On the Feller strong law of large numbers for fields of B-valued random variables. *Math. Pannonica*, 5 (1994), 249–254 .

[77] Z.A. Łagodowski, Z. Rychlik, Rates of convergence in the weak law for martingale random fields, *Teor. Veroyatn. Mat. Stat./Theory Probab. Math. Stat.*, 2 (1993), 119–130.

[78] Z.A. Łagodowski, Fuk's inequalities for stochastic fields of reversed martingales, *Sib. Math. J.*, 32 (1991), 176–179.

[79] Z.A. Łagodowski, Z. Rychlik, Weak convergence of probability measures on the function space $D_d[0, \infty)$, *Bull. Pol. Acad. Sci., Math.*, 34 (1986), 329–335.

[80] Z.A. Łagodowski, Z. Rychlik, Rates of convergence in the weak law for martingale random fields, *T*heory Probab. Math. Stat., 2 (1993), 119–130.

[81] Z.A. Łagodowski, P. Matuła, On almost sure limiting behavior of weighted sum of random fields, Acta Math. Hung., 126 (2010), 16–22.

[82] Z.A. Łagodowski, P. Matuła, SLLN for random fields under conditions on the bivariate dependence structure, Publ. Math. Debrecen, 76 (2010), 329–339.

[83] Z.A. Łagodowski, On the necessary condition for Baum-Katz type theorem for non-identically distributed and negatively dependent random fields, *Ann. Univ. Maria Curie-Skłodowska Sect. A,* 72 (2018), 1–8.

[84] A. Kuczmaszewska, Z.A. Łagodowski, Convergence rates in the SLLN for some classes of dependent random fields, *J. Math. Anal. Appl.*, 380 (2011), 571–584.

[85] Z.A. Łagodowski, An approach to complete convergence theorems for dependent random fields via application of Fuk–Nagaev inequality, *J. Math. Anal. Appl.*, 437 (2016), 380–395.

[86] Z.A. Lagodowski, Strong laws of large numbers for $\mathbb{B}$-valued random fields, *Discrete Dyn. Nat. Soc.*, (2009) DOI: https://doi:10.1155/2009/485412

[87] J. Li, On pairwise quasi-asymptotically independent random variables and their applications, *Statist. Probab. Lett.*, 83 (2013), 2081–2087.

[88] A.I. Martikainen, Order of growth of a random field, *Math. Notes*, 39 (1986), 431–437.

[89] A. Martikainen, On the asymptotic behavior of sums of pairwise independent random variabies, *Statist. Probab. Lett.*, 25 (1995), 21–26.

[90] P. Matuła, A note on the almost sure convergence of sums of negatively dependent random variables, *Statist. Probab. Lett.*, 15 (1992), 209–213.

[91] P. Matuła, On some families of AQSI random variables and related strong law of large numbers, *Appl. Math. E-Notes*, 5 (2005), 31–35.

[92] E. Merzbach, Stopping for two-dimensional stochastic processes, *Stochastic Processes Appl.*, 10 (1980), 49–63.

[93] E. Merzbach, Extension and continuity of the stochastic integral in the plane, *Notes Sem. Proc. Aleatoires et Applic. Telecommunications*, 5 (1981), 79–88.

[94] E. Merzbach, M. Zakai, Stopping a two-parameter weak martingale, *Probab. Theory Relat. Fields*, 76 (1987), 499–507.

[95] T. Mikosh, On the strong law of large numbers for random fields, *Vestnik Leningrad. Univ., ser. Mat., Mech., Astr.*, 19 (1984), 82–85.

[96] T. Mikosh, R. Norvaisa, Strong laws of large numbers for fields of Banach space valued random variables, *Probab. Theory Relat. Fields*, 74 (1987), 241–253.

[97] A. Millet, L. Sucheston, Characterization of Vitali conditions with overlap in terms of convergence of classes of amarts, *Can. J. Math.*, 31 (1979), 1033–1046.

[98] A. Millet, L. Sucheston, A Characterization of Vitali Conditions in Terms of Maximal Inequalities, *Ann. Probab.*, 8 (1980), 339–349.

[99] F.C. Móricz, U. Stadtmüller, Summability of double sequences by weighted mean methods and Tauberian conditions for convergence in Pringsheim's sense, *Int. J. Math. Math. Sci.*, 65 (2004), 3499–3511.

[100] F.C. Móricz, Tauberian theorems for Cesàro summamble double sequences, *Stud. Math.*, 110 (1994), 83–96.

[101] R.B. Nelsen, An introduction to copulas, Springer-Verlag, New York (1999).

[102] J. Neveu, Discrete Parameter Martingales, revised ed., North-Holland Publishing Co., Amsterdam (1975).

[103] M. Peligrad, Maximum of partial sums and an invariance principle for a class of weakly dependent random variables, *Proc. Amer. Math. Soc.*, 126 (1998), 1181–1189.

[104] M. Peligrad, A. Gut, Almost-sure results for class of dependent random variables, *J. Theor. Probab.*, 12 (1999), 87–104.

[105] V. V. Petrov, A note on the Borel–Cantelli lemma, *Statist. Probab. Lett.*, 58 (2002), 283–286

[106] A.R. Pruss, Randomly sampled Riemann sums and complete convergence in the law of large numbers for a case without identical distribution, *Proc. Amer. Math. Soc.*, 124 (1996) 919–929.

[107] A. Rosalsky, L.V. Than, Strong and weak laws of large numbers for double sums of independent random elements in Rademacher type p Banach spaces, *Stochastic Anal. Appl.*, 24 (2006), 1097–1117.

[108] G. Roussas, Asymptotic normality of random fields of positively or negatively associated processes, *J. Multivariate Anal.* 50 (1994), 152–173.

[109] Q.M. Shao, A comparison theorem on moment inequalities between negatively associated and independent random variables, *J. Theor. Probab.*, 13 (2000), 343–356.

[110] G.R. Shorack, R.T. Smythe, Inequalities for $\max|S_k|/b_k$ where $\mathbf{k} \in \mathbb{N}^d$, *Proc. Amer. Math. Soc.*, 54 (1976), 331–336.

[111] R.T. Smythe, Strong law of large numbers for r-dimensional arrays of random variables, *Ann. Probab.*, 1 (1973), 164–170.

[112] R. Smythe, Sums of independent random variables on partially ordered sets, *Ann. Probab.*, 2 (1974), 906–917.

[113] T.C. Son, D.H. Thang, The Brunk-Prokhorov strong law of large numbers for fields of martingale differences taking values in a Banach space, *Statist. Probab. Lett.*, 83 (2013), 1901–1910.

[114] S.H. Sung, On Complete convergence for arrays of dependent random variables, *Commun. Stat., Theory Methods*, 41 (2012), 1663–1674.

[115] S.H. Sung, A.I. Volodin, T.-Ch. Hu, More on complete convergence for arrays, *Statist. Probab. Lett.*, 71 (2005), 303–311.

[116] J.E. Taylor, K.J. Worsley, Random fields of multivariate test statistics, with applications to shape analysis, *Ann. Stat.,* 36 (2008), 1–27.

[117] J.B. Walsh, Martingales with a multidimensional parameter and stochastic integrals in the plane. *Lectures in probability and statistics*, Springer (1986), 329–491.

[118] M. Wichura, Inequalities with applications to the weak convergence of Random processes with multi-dimensional time parameters, *Ann. Math. Stat.*, 40 (1969), 681–687.

[119] N. Wiener, The ergodic theorem, *Duke Math. J.*, 5 (1939), 1–18.

[120] E. Wong, M. Zakai, Weak martingales and stochastic integrals in the plane, *Ann. Probab.*, 4 (1976), 570–586.

[121] E. Wong, M. Zakai, Martingales and stochastic integrals for processes with a multidimensional parameter, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, 29 (1974), 109–122.

[122] K.J. Worsley, J. Cao, T. Paus, M. Petrides, A.C. Evans, Applications of random field theory to functional connectivity, *Hum. Brain Mapping*, 6 (1998), 364–367.

[123] K.J. Worsley, K.J. Friston, A test for a conjunction, *Statist. Probab. Lett.*, 47 (2000), 135–140.

[124] K.J. Worsley, Brain mapping data: classification, principal components and multidimensional scaling, *New Developments in Psychometrics*, Proceedings of the International Meeting of the Psychometric Society, Springer, Japan 2003, 585–592.

[125] W.A. Woyczynski, On Marcinkiewicz strong law of large numbers in Banach spaces and related rates of convergence, *Probab. Math. Stat.*, 1 (1980), 117–131.

[126] M. Zakai, Some classes of two-parameter martingales, *Ann. Probab.*, 9 (1981), 255–265.

[127] L. Zhang, Y. Miao, J. Mu, J. Xu, Complete convergence for weighted sums of mixingale sequences and statistical applications, *Commun. Stat., Theory Methods* 46 (2017), 10692–10701.

[128] L.X. Zhang, J. Wen, A weak convergence for negatively associated fields, *Statist. Probab. Lett.*, 53 (2001), 259–267.

[129] G., Zimmerman, Some sample function properties of the two parameter Gaussian process, *Ann. Math. Statist.*, 43,(1972) 1235–1246.

[130] A. Zygmund, An individual ergodic theorem for non-commutative transformations, *Acta Sci. Math.* (Szeged), 14 (1951), 103–110.

[131] X. Yang, K.-A. Fu, A Hoffmann–Jørgensen inequality of NA random variables with applications to the convergence rate, *Math. Inequal. Appl.*, 16 (2013), 313–328.